Unitary Manifold Restoration and the Spectral Topology of the Riemann Zeta Function

A Complete Obstruction Classification for the Riemann Hypothesis

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Abstract

This paper presents the Unitary Manifold Restoration (UMR) framework and a structural analysis of the Riemann Hypothesis obstruction. We prove the following unconditional results: the **Viète Convergence Threshold** ($\mathcal{E}(\sigma) < \infty$ iff $\sigma > 1/2$); the **Near-Zone Detection Theorem** (any failure of monotonicity in the near zone localizes an off-line zero within an explicit disk, with quantitative depth control; monotonicity is therefore unconditional at all heights where RH is verified, and holds under RH in general), with corrected coverage quantification; the **Sign Partition Lemma** (negativity of a pair-geometry contribution requires $\beta_k > \sigma$ strictly); the **Cascade Theorem** ($v_1(\sigma, \gamma) > 0$ for $\sigma > \sigma^*$ combined with the Vinogradov–Korobov zero-free region, with no circularity); the **Pair Geometry Theorems** (negative-contribution window, integral identity, single-pair dominance); the **Explicit-Formula Bridge** ($\psi(x) - x = -2\sqrt{x}\,\mathrm{Re}\,\widetilde{S}(x,T) + O(x\log^2 x/T)$, empirically verified against the first 300 zeros; the coherence bound $A < 2$ implies RH unconditionally); the **Supply-Demand Obstruction** and Gallagher gap quantification; and the **Universal Obstruction Theorem** (every classical approach reduces to one of two irreducible obstructions, A or B).

We also establish twenty-two Closed Channels, classifying twenty-two independent approaches as impassable (twenty by undershoot, two by overshoot: de Branges positivity via Conrey–Li, and the bounded-supremum/Bohr–Jessen-support channel via Kronecker alignment). Key results include: the **Shell Concentration Theorem** (pointwise form: at most $O(\log \gamma_0)$ zeros can contribute negatively at any evaluation point, each with damage bounded by the reciprocal shell depth $1/(v_k - \delta_0)$; a single shallow zero realizes the obstruction); the **Duality of Obstructions** (Theorem 10.3: the two irreducible obstructions are dual faces of the same singularity); the L^1 **Neutrality Theorem** (each off-line pair is exactly L^1-neutral: integrated methods are blind to displacement, so the obstruction is irreducibly pointwise); and a **computational test** for the shell concentration (Section 4.2). The complete obstruction catalogue is the paper's central contribution.

We also establish and verify the **On-Line Residual Distributional Law**: GUE-conditionally, the on-line residual slope satisfies $\mathbb{E}[R(\gamma)] = \frac{1}{6}\log^2(\gamma/2\pi)$ with a

$t^{-3/2}$ upper tail and an $e^{-c/r}$ margin floor; the tail and the variance convergence are *measured* against Odlyzko's table of the first two million zeros, with all predictions registered in advance of the data (four hits, one directional, one miss in the rigid direction, scorecard preserved).

This paper does not contain a proof of the Riemann Hypothesis. The single open problem ($v_1(\sigma, \gamma) > 0$ for all $\sigma > 1/2$) is equivalent to RH — a reformulation, not a simplification.

Contents

1 Introduction

The Riemann Hypothesis (RH) asserts that every non-trivial zero of $\zeta(s)$ satisfies $\mathrm{Re}(s) = 1/2$. This paper does not prove RH. Instead, we develop the UMR framework to answer a precise question: *why does every classical approach fail?*

The answer is a theorem: every approach reduces to one of two irreducible obstructions (Theorem 10.1). Moreover, these two obstructions are dual faces of the same analytic singularity (Theorem 10.3).

Executive Summary (for the busy reader).

- **Unconditional results:** Near-zone detection theorem, Sign Partition Lemma, Cascade Theorem (far-zone strip), Shell Concentration (pointwise localization), Viète threshold, explicit-formula bridge, Supply-Demand obstruction, Gallagher gap (≈ 2500).
- **Universal obstruction:** Every classical method reduces to (A) PNT in intervals shorter than $x^{7/12}$ or (B) the Cauchy–Stieltjes singularity at $v = \delta^+$. These are the same obstruction (dual).
- **Open problem:** $v_1(\sigma, \gamma) > 0$ for σ in the central gap. Equivalent to RH.
- **This paper does not prove RH.** It completely classifies why existing methods fail.

Summary of unconditional results

(i) **Viète Convergence Threshold (§8):** $\mathcal{E}(\sigma) < \infty$ iff $\sigma > 1/2$.

(ii) **Near-Zone Detection (§4):** If $|\zeta(\sigma + i\gamma)|$ fails to be strictly monotone for $|\sigma - 1/2| < 2/(\log\gamma + 2C_0)$, an off-line zero lies within an explicit disk about the evaluation point, with quantitative depth localization. Monotonicity is unconditional for all $\gamma \leq 3 \times 10^{12}$ [12] and holds under RH in general. Coverage of the half-strip: $\approx 89\%$ at $\gamma = 14$, $\approx 25.6\%$ at $\gamma = 10^6$, decaying as $O(1/\log\gamma)$.

(iii) **Sign Partition Lemma (§4):** $P_k(\delta) < 0 \iff \beta_k > \sigma$ strictly.

(iv) **Cascade Theorem (§5):** Combined with the Vinogradov–Korobov zero-free region (no zeros with $\beta > 1 - c(\log\gamma)^{-2/3}(\log\log\gamma)^{-1/3}$; explicit constants in [18]), $v_1(\sigma, \gamma) > 0$ for all $\sigma > 1 - c(\log\gamma)^{-2/3}(\log\log\gamma)^{-1/3}$, without circularity. Together with the near-zone detection theorem, this gives one unconditional v_1-positivity strip (far zone) and a detection-controlled near zone, with a central gap between them.

(v) **Pair Geometry Theorems (§6):** The integral of an off-line pair's v_1 contribution over all heights vanishes identically for $\delta \in (0, v^*)$ (Pair Integral Identity, Theorem 6.1); the pair is exactly L^1-neutral (Theorem 6.7), so integrated methods cannot see displacement; the negative-window integral is $4\arctan\sqrt{(v^* - \delta)/(v^* + \delta)} - \pi$ (Proposition 6.3); the negative-contribution window is the disk $\delta^2 + \Delta^2 < v^{*2}$ (Theorem 6.4); for $\delta < v^*/\sqrt{3}$, v_3 dominates a single off-line pair's maximum negative contribution (Theorem 6.5); and the pointwise obstruction is governed by the minimal shell depth among $O(\log \gamma_0)$ local candidates (Theorem 6.11).

(vi) **Explicit-Formula Bridge (§9):** $\psi(x) - x = -2\sqrt{x}\,\mathrm{Re}\,\widetilde{S}(x,T) + O(x \log^2 x/T)$, unconditional and empirically verified; Conjecture 7.6 ($A < 2$) implies RH, while the unconditional prime-counting error stands a factor $\exp((\log x)^{2/5+o(1)})$ from what the conjecture requires.

(vii) **Supply-Demand Obstruction and Gallagher Gap (§7):** No depletion-based argument reaches $\sigma_0 = 1/2 + \varepsilon$ for small ε. At $\log T = 1000$: worst-case to typical coherence ratio $\approx 2{,}500$.

(viii) **Universal Obstruction Theorem (§10):** Every classical approach reduces to one of two irreducible obstructions: (A) the Huxley–PNT gap, or (B) the Cauchy–Stieltjes singularity at $v = \delta^+$. These are the same obstruction in different coordinates.

What this paper does not do

We do not prove RH. Conjecture 4.14 is equivalent to RH, not a reduction. The twenty-two closed channels in §13 and the Universal Obstruction Theorem constitute a *completeness theorem for obstruction classification*: any future proof of RH must supply an ingredient not present in any of those twenty-two channels.

Guide for the reader. Section 2 establishes the unitarity characterization. Section 3 defines the three-force decomposition with explicit formulas for v_1, v_2, and v_3. Sections 4–5 establish the near-zone detection theorem, the far-zone v_1-positivity strip, and the localization of any obstruction to a thin shell about the evaluation abscissa. Sections 6–9 develop the pair geometry, supply-demand analysis, and Guinand–Weil equivalence. Sections 10–11 show that every classical approach reduces to one of two irreducible obstructions, catalogued in Section 13. Section 12 concludes with the equivalence to RH.

Relation to prior work

The obstruction classification developed here complements, rather than competes with, several influential lines of research on the Riemann zeta function. Conrey, Farmer, and Zirnbauer [36] developed n-level correlation predictions based on random matrix models, yielding precise conjectures for moments and pair correlations of zeros; that framework operates at the probabilistic-ensemble level and does not resolve the sign question for individual zero heights. Soundararajan [22] and Radziwiłł–Soundararajan [23] established sharp large-value estimates for $\log \zeta(\frac{1}{2} + it)$ and related quantities, yielding bounds on moment sums via resonance methods; these are moment-averaged results and are covered by Closed Channel C6 and the Second Moment Ceiling (Closed Channel 9.5). Levinson [5] and Conrey [6] proved unconditionally that positive proportions of zeros lie on the critical line using the mollifier method and Dirichlet polynomial approximations; such results establish density on the line but do not exclude finitely many or measure-zero off-line zeros, the configuration that the three-force decomposition targets. Each of these approaches operates at a different level of analytic structure than the present sign-based, three-force decomposition. Independently of this work, Islam [43, 44] has developed a parity-based obstruction localization for RH: the functional equation acts on the critical line as the reflection $\tau \mapsto -\tau$, every identity it generates is parity-even, and the off-line signature survives only in the odd (imaginary) channel — his on-line pair-cancellation computation is the boundary-value form of our Lemma 4.11 and of the L^1 Neutrality Theorem (Theorem 6.7). The two programs converge on the same structural conclusion (symmetric and averaged functionals are provably blind to real-part displacement; the obstruction is a single object equivalent to RH; closure requires Euler-product arithmetic input) by disjoint methods, his qualitative parity/Hardy-space localization complementing the quantitative obstruction measurement developed here. The classical witness for the insufficiency of functional-equation data alone is the Davenport–Heilbronn function [41, 42]: it satisfies an exact Riemann-type functional equation yet possesses zeros off the critical line, so membership in the Euler-product subclass is precisely the additional input any proof must use. Consequently, our obstruction classification does not subsume or invalidate those results; it provides a complementary lens showing that any future proof must introduce arithmetic or spectral input genuinely absent from all twenty-two catalogued channels. For surveys of classical and modern methods, see [2, 31].

2 Unitarity and Basin Geometry

Theorem 2.1 (Unitarity Characterization, $|t| \geq 7$)**.** *Let $\chi(s) = 2^s \pi^{s-1} \sin(\pi s/2)\,\Gamma(1-s)$ and $s = \sigma + it$ with $0 < \sigma < 1$ and $|t| \geq 7$. Then $|\chi(s)| = 1$ if and only if $\sigma = 1/2$.*

Proof. On the line: $\chi(s)\chi(1-s) = 1$ with $1-s = \bar{s}$ and the Schwarz reflection $\chi(\bar{s}) = \overline{\chi(s)}$ give $|\chi|^2 = 1$ exactly. Off the line: $\partial_\sigma \log|\chi(\sigma+it)| = \mathrm{Re}[\chi'/\chi](\sigma+it) = -\log(t/2\pi) + O(1/t^2)$, which is strictly negative throughout $0 < \sigma < 1$ for $|t| \geq 7$ (verified numerically across the strip: the maximum over σ of $\mathrm{Re}\,\chi'/\chi$ is -0.107 at $t = 7$ and decreases thereafter). Hence $|\chi(\sigma+it)|$ is strictly decreasing in σ at each such height and equals 1 only at $\sigma = 1/2$. □

Remark 2.2 (The height restriction is necessary)**.** *The level set $\{|\chi(s)| = 1\}$ contains off-line points: direct computation gives $|\chi(0.75+it)| = |\chi(0.25+it)| = 1$ at $t \approx 6.2882$, where the level curve crosses the strip horizontally near $t = 2\pi$ and $\mathrm{Re}\,\chi'/\chi(1/2+it)$ changes sign. Versions through v6.8 stated the characterization without the restriction; since every nontrivial zero height satisfies $\gamma \geq \gamma_1 > 14$, all downstream uses are unaffected.*

Theorem 2.3 (Conical Profile, $t > 2\pi$)**.** *For small δ and $t > 2\pi$, $\big||\chi(\tfrac{1}{2}+\delta+it)|^2 - 1\big| = C(t)|\delta| + O(\delta^2)$ where $C(t) = 2\,|\mathrm{Re}\,\frac{d}{ds}\log\chi(s)|_{s=1/2+it}| = 2\log(t/2\pi) + O(1/t^2) > 0$. At $t = 2\pi$ the coefficient vanishes (*$\mathrm{Re}\,\chi'/\chi$ *changes sign there), and for $t < 2\pi$ the basin orientation inverts; the conical geometry holds at all zero heights since $\gamma_1 > 14 > 2\pi$.*

Remark 2.4. *This theorem describes the behavior of $|\chi(s)|$, not of $\zeta(s)$. It does not by itself constrain the location of zeros.*

3 The Three-Force Decomposition

Theorem 3.1 (Monotonicity Equivalence, ξ-form)**.** *The following are equivalent: (i) RH holds; (ii)* $\mathrm{Re}\,[\xi'/\xi](s) > 0$ *for every* s *with* $\mathrm{Re}(s) > 1/2$*; (iii)* $\sigma \mapsto |\xi(\sigma + it)|$ *is strictly increasing on* $(1/2, \infty)$ *for every* $t \in \mathbb{R}$ *(Sondow–Dumitrescu [39]).*

Proof. (i)⇒(ii): by the Hadamard product, $\mathrm{Re}\,\xi'/\xi(s) = \sum_\rho \mathrm{Re}\,\frac{1}{s-\rho}$ (zeros paired with their reflections); under RH every term equals $\delta/(\delta^2 + (t-\gamma_k)^2) > 0$ for $\delta > 0$. (ii)⇒(i): a zero $\rho^* = \beta + i\gamma^*$ with $\beta > 1/2$ forces $\mathrm{Re}\,\xi'/\xi(\sigma + i\gamma^*) \to -\infty$ as $\sigma \to \beta^-$, violating (ii) at points with $\sigma > 1/2$. (ii)⇔(iii) is the definition of strict monotonicity of $\log|\xi|$ in σ. □

Lemma 3.2 (Exact ξ-identity for the decomposition)**.** *At every zero height* γ *and* $\sigma > 1/2$*,*

$$\mathrm{Re}\,[\xi'/\xi](\sigma + i\gamma) \;=\; v_1(\sigma, \gamma) + v_3(\sigma, \gamma)$$

exactly: the background v_2 *of Lemma 3.4 is precisely the* ζ*-to-*ξ *correction,* $v_2 = \mathrm{Re}[\zeta'/\zeta - \xi'/\xi] = \frac{1}{2}\log\pi - \frac{1}{2}\mathrm{Re}\,\psi(s/2) - \mathrm{Re}\frac{1}{s} - \mathrm{Re}\frac{1}{s-1}$*. The completed function carries no background: the open problem (Conjecture 4.14) is the positivity of* $\mathrm{Re}\,\xi'/\xi$ *beyond the self-zero term.*

Proof. $\xi(s) = \frac{1}{2}s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s)$ gives $\xi'/\xi = \zeta'/\zeta + 1/s + 1/(s-1) - \frac{1}{2}\log\pi + \frac{1}{2}\psi(s/2)$, whose real part is $L - v_2$ with the regularized v_2 of Lemma 3.4. Since $L = v_1 + v_2 + v_3$, the identity follows. (This is also the numerical verification of v6.8 read structurally: the corrected v_2 is exactly the term the completed function removes.) □

Remark 3.3 (The $|\zeta|$ form is false — computational refutation)**.** *Versions through v6.8 stated the equivalence for* $|\zeta|$*: that under RH,* $\sigma \mapsto |\zeta(\sigma + i\gamma)|$ *is strictly increasing on* $(1/2, 1)$ *at every on-line zero height. This is refuted by direct computation at heights where RH is verified [12]:* $L(\sigma, \gamma) = \mathrm{Re}\,\zeta'/\zeta < 0$ *occurs at* 15 *of* 250 *consecutive zero heights near* $\gamma \approx 4500$ *at* $\sigma = 0.99$*, with minimum* $L = -0.15410$ *at* $\gamma_{3941} = 4449.6353$ *(confirmed at* 30*-digit precision), each failure at a locally sparse gap configuration. The mechanism is the background:* $v_2 \approx -\frac{1}{2}\log(\gamma/2\pi)$ *grows without bound while* $v_3 \le 1/\delta$ *is bounded for fixed* δ*, so* $|\zeta|$*-monotonicity fails under RH whenever* v_1 *fluctuates low, while* $|\xi|$*-monotonicity, carrying no background by Lemma 3.2, is exactly equivalent to RH.*

Lemma 3.4 (Three-Force Decomposition)**.** *At* $s = \sigma + i\gamma$ *with* $\delta = \sigma - 1/2 \neq 0$ *and* $\zeta(1/2 + i\gamma) = 0$*:*

$$L(\sigma) := \mathrm{Re}\left[\frac{\zeta'}{\zeta}(\sigma + i\gamma)\right] = v_1(\sigma, \gamma) + v_2(\sigma, \gamma) + v_3(\sigma, \gamma),$$

where the three components are defined explicitly as follows.

The volatile term v_1 *is the aggregate contribution of all other nontrivial zeros:*

$$v_1(\sigma, \gamma) = \sum_{\substack{\rho_k = \beta_k + i\gamma_k \\ \gamma_k \neq \gamma}} \frac{\sigma - \beta_k}{(\sigma - \beta_k)^2 + (\gamma - \gamma_k)^2},$$

where the sum runs over all nontrivial zeros ρ_k of ζ (counted with multiplicity) and converges absolutely for $\sigma > 1/2$ by the Weierstrass–Hadamard product representation. Under RH, $\beta_k = 1/2$ for all k, and $v_1 = \sum_{\gamma_k \neq \gamma} \delta/(\delta^2 + (\gamma - \gamma_k)^2) > 0$.

The background term *v_2 collects the pole at $s = 1$, the prefactor, and the Gamma-factor contribution, in the regularized Hadamard–ξ form:*

$$v_2(\sigma, \gamma) = \tfrac{1}{2} \log \pi - \tfrac{1}{2} \mathrm{Re}\, \psi\big(\tfrac{s}{2}\big) - \mathrm{Re}\, \frac{1}{s} - \mathrm{Re}\, \frac{1}{s-1},$$

where $\psi = \Gamma'/\Gamma$ is the digamma function; the trivial zeros are carried inside the digamma term. (An earlier form of this lemma, retained through v6.7, wrote the trivial-zero contribution as a separate series $\sum_n (\sigma + 2n)/((\sigma + 2n)^2 + \gamma^2)$ with the digamma term of opposite sign; that series diverges and the formula is retracted — see Remark 4.4.) The decomposition $L = v_1 + v_2 + v_3$ with the regularized v_2 has been verified numerically at zero heights $\gamma \approx 79$–473 across $\delta \in [0.05, 0.3]$, with residuals at the 10^{-3} level fully accounted for by truncation of v_1. This v_2 satisfies $|v_2| \le \frac{1}{2} \log \gamma + C_0$ with $C_0 \approx 0.92$ (Backlund's constant, Titchmarsh Theorem 9.6(A)), and is nearly flat in σ at fixed large γ: $v_2(\sigma, \gamma) - v_2(\frac{1}{2}, \gamma) = O(\delta/\gamma)$.

The self-interaction term *v_3 is the contribution of the on-line zero at height γ itself:*

$$v_3(\sigma, \gamma) = \frac{1}{\delta} + \frac{\delta}{\delta^2 + 4\gamma^2},$$

which has $\mathrm{sign}(\delta)$ and dominates the decomposition in the near-zone.

4 Zero Confinement

4.1 Near-Zone (Unconditional)

Theorem 4.1 (Near-Zone Detection). *Let γ be a zero height and $w(\gamma) = 2/(\log \gamma + 2C_0)$. Unconditionally, for every $\delta \in (0, w(\gamma))$, exactly one of the following holds:*

1. *$L(1/2 + \delta, \gamma) > 0$: $|\zeta(\sigma + i\gamma)|$ is strictly increasing at $\sigma = 1/2 + \delta$ (and, by the functional equation, strictly decreasing at $1/2 - \delta$); or*

2. *an off-line zero $\rho_k = \beta_k + i\gamma_k$ exists with $\beta_k > 1/2 + \delta$ and $\delta^2 + (\gamma - \gamma_k)^2 < (\beta_k - 1/2)^2$, and at least one such zero has shell depth*

$$d_k \;=\; \beta_k - \tfrac{1}{2} - \delta \;\le\; \frac{\kappa \log \gamma}{1/\delta - \frac{1}{2} \log \gamma - C_0}.$$

In particular: (a) any failure of near-zone monotonicity localizes an off-line zero within an explicit disk, with quantitative depth control; (b) for all $\gamma \le 3 \times 10^{12}$, monotonicity holds unconditionally throughout the near zone [12]; (c) under RH, near-zone monotonicity holds at every height. The near-zone width covers $\approx 89\%$ of the half-strip $(1/2, 1)$ at $\gamma = 14$ and $\approx 25.6\%$ at $\gamma = 10^6$, decaying as $O(1/\log \gamma)$.

Proof. For $\delta \in (0, w(\gamma))$, $v_3 \geq 1/\delta$ and $|v_2| \leq \frac{1}{2}\log\gamma + C_0 < 1/\delta$, so $M(\delta) := v_3 - |v_2| \geq 1/\delta - \frac{1}{2}\log\gamma - C_0 > 0$, and $L \leq 0$ forces $v_1 \leq -M(\delta) < 0$. By the Sign Partition Lemma (Proposition 4.13), $v_1 < 0$ requires at least one zero with $\delta^2 + (\gamma - \gamma_k)^2 < v_k^2$; any such zero is off-line with $\beta_k > 1/2 + \delta$ and $|\gamma_k - \gamma| < v_k < 1/2$. By Lemma 6.10, at most $\kappa \log\gamma$ zeros satisfy the height constraint, and each contributes at least $-1/d_k$ (Theorem 6.11(2)); hence $\kappa \log\gamma / d_{\min} \geq M(\delta)$, which is the stated depth bound. Statement (b) follows from [12]: below height 3×10^{12} there are no off-line zeros, hence none in the disk, hence alternative (1) holds throughout. □

Remark 4.2 (Why the v6.7 statement was too strong)**.** *Versions through v6.7 asserted unconditional strict monotonicity outright, with the proof "$|v_3| > |v_2|$, so $v_2 + v_3$ has* sign(δ) *regardless of v_1." The conclusion requires $L = v_1 + v_2 + v_3 > 0$, and nothing unconditional prevents $v_1 < -(v_2 + v_3)$: a direct computation at $\gamma_{101} \approx 237.77$ shows that a single hypothetical off-line pair at $v^* = 0.6\,w$, height offset 0.002, flips the sign of L at $\delta \in (0.40\,w,\, 0.60\,w)$ — strictly inside the claimed zone. Excluding such configurations is locally equivalent to RH. The detection form above is the correct unconditional content; it converts every potential failure into a falsifiable localization statement, in exact agreement with the pointwise Shell Concentration Theorem.*

Remark 4.3 (On-Line Residual — Retracted)**.** *Versions through v6.7 asserted that $S_{\mathrm{on}}(\delta, \gamma) + v_2(\sigma, \gamma) - v_2(\frac{1}{2}, \gamma) = \delta \log(2\pi) + O(\delta/\gamma)$, a universal residual. Direct computation against the actual zeros refutes this. In the near zone the true behaviour is*

$$S_{\mathrm{on}}(\delta, \gamma) + \left[v_2(\sigma, \gamma) - v_2(\tfrac{1}{2}, \gamma)\right] = \delta \cdot R(\gamma) + O(\delta^2), \qquad R(\gamma) = 2\sum_{\gamma_k \neq \gamma} \frac{1}{(\gamma - \gamma_k)^2},$$

a height-dependent, configuration-sensitive quantity: computed values of R at six zero heights $\gamma \in [104, 473]$ are 1.56, 1.85, 2.74, 3.62, 2.49, 3.52, fluctuating with the local zero configuration. The earlier claim matched the data at one height (γ_{51}, $R = 1.848 \approx \log 2\pi = 1.838$) by coincidence. The retracted derivation also used $v_2(\sigma) - v_2(\frac{1}{2}) = -\delta \log(\gamma/2\pi)$; in fact the regularized background is nearly flat in σ, with difference $O(\delta/\gamma)$ (Lemma 3.4). Whether $R(\gamma)$ admits a clean distributional law (e.g., via GUE pair-correlation statistics) is an open question noted for future work.

Remark 4.4 (Corrigendum: v6.7 → v6.8)**.** *Three further corrections, following direct numerical verification against the actual zeros (June 2026): (i) the v_2 formula in Lemma 3.4 contained a divergent trivial-zero series and inverted sign structure; the regularized Hadamard–ξ form above is verified to truncation accuracy; (ii) the On-Line Residual proposition ($\delta \log 2\pi$, universal) is retracted — the true residual is height-dependent (Remark 4.3); (iii) Near-Zone Monotonicity, asserted unconditionally through v6.7, is restated as the Near-Zone Detection Theorem: the original proof established only $v_2 + v_3 > 0$, which does not bound v_1 from below, and a computed counterexample configuration breaks monotonicity strictly inside the zone. Coverage figures are corrected (89% at $\gamma = 14$, not 100%).*

Remark 4.5 (Corrigendum: v6.8 → v6.9)**.** *Three further corrections, all from direct computation (June 2026): (i) the Unitarity Characterization required the height restriction*

$|t| \geq 7$ *— the level set* $|\chi| = 1$ *contains off-line points at* $t \approx 6.2882$ *near* $t = 2\pi$*, where* $\operatorname{Re}\chi'/\chi$ *changes sign on the line; the Conical Profile is restricted to* $t > 2\pi$ *accordingly (all zero heights satisfy* $\gamma > 14$*, so no downstream use is affected); (ii) the Monotonicity Equivalence, stated for* $|\zeta|$ *through v6.8, is false in its forward direction —* $L < 0$ *occurs at verified zero heights (Remark 3.3) — and is restated exactly for the completed function* ξ*, where it is classical [39]; (iii) the accompanying identity* $\operatorname{Re}\xi'/\xi = v_1 + v_3$ *(Lemma 3.2) shows the v6.8-corrected background* v_2 *is precisely the* ζ*-to-*ξ *correction, so the three-force decomposition is, structurally, the Hadamard sum of the completed function plus an explicitly known background.*

Remark 4.6 (Corrigendum: v6.9 → v6.10)**.** *Three corrections from the final verification pass (June 2026): (i) the Guinand–Weil short-interval equivalence is retracted — its error term was pointwise impossible and its weights mismatched — and replaced by the unconditional, empirically verified Explicit-Formula Bridge (Proposition 9.1); the proof that the coherence bound implies RH is now self-contained via Landau's theorem. (ii) The coherence conjecture was overlisted as* equivalent *to RH; it implies RH, while RH returns only the exponent* $A = 2$ *(von Koch), so it is removed from the equivalence list of Theorem 12.2 and restated as a one-way implication. (iii) The equivalence (i)⇔(ii) of Theorem 12.2, previously asserted, is now proved under the explicit zero-height convention of Definition 12.1. The Supply-Demand crossover table and the Gallagher gap figures were verified by direct computation (all entries exact).*

Remark 4.7 (Corrigendum: v6.10 → v6.11)**.** *Three items from the Halász session (June 10, 2026). (i) The Gap Channel Supremum Conjecture of the companion research record (master document, §10 insert of May 13, 2026) —* $\sup_t |\log|\zeta(\sigma + it)|| < \infty$ *for fixed* $\sigma > 1/2$ *— is* refuted unconditionally *and is recorded here as Closed Channel C22 (Section 13). The windowed Coherence Bound (Conjecture 7.6) is a different object and is unaffected; see Remark 7.7. (ii) Citation correction: the Dirichlet-polynomial large-values inequality used in that lineage is the Halász–Montgomery inequality (Montgomery 1969; Iwaniec–Kowalski Thm. 9.6), not Halász 1971, which is the multiplicative mean-value theorem. (iii) One stale channel count ("twenty") surviving from v6.9 in the introductory subsection "Relation to prior work" is corrected to the current total.*

Remark 4.8 (Corrigendum: v6.11 → v6.12)**.** *The* $(\log\gamma)^4$ *aggregate-drain heuristic, introduced and explicitly flagged as non-theorem in v6.11, is refuted by the Drain Ceiling (Lemma 13.4), an immediate consequence of the Sign Partition Lemma already in the paper. The neighbor-drain remark is restated with the exact second-order law and its drain/boost sign boundary at* $g \approx a/\sqrt{3}$*, which the v6.11 test grid (all* $g \gg a$*) could not detect. The v6.11 statement that the sign was "uniformly negative at all tested* δ, σ, g*" was accurate for its stated range and required refinement, not retraction.*

Remark 4.9 (Corrigendum: v6.12 → v6.13)**.** *The On-Line Residual distributional law (Section 4.4) is added: exact GUE-conditional mean* $\mathbb{E}[R(\gamma)] = \frac{1}{6}\log^2(\gamma/2\pi)$*,* $t^{-3/2}$ *upper tail,* $e^{-c/r}$ *margin floor, empirical verification at zeros* 20–600 *and* 4925–5075*, and the migration-site census. One audit flag: the figure* $\operatorname{Var}[v_1] \approx 0.075$ *at* $\gamma \sim 550$ *quoted in*

Remark 4.16 did not reproduce under self-consistent recomputation (obtained 0.021–0.035; see Remark 4.21); the original measurement's window and script are to be recovered, and the figure should not be cited until resolved. An intermediate draft of the new section reported a variance convergence plateau based on that figure; the claim was superseded within the same working day by the self-consistent three-point curve.

4.2 A Computational Test for the Shell Concentration

Remark 4.10 (Computational test of the obstruction). *The pointwise Shell Concentration Theorem (Theorem 6.11) identifies the only dangerous configuration: a hypothetical zero of shallow depth $v_k - \delta_0 < (\sqrt{3}-1)\delta_0$ at nearly matching height. Rigorous verification of RH up to height 3×10^{12} [12] excludes off-line zeros entirely at computationally accessible heights, so the operative test is not shell detection but direct evaluation of v_1 at zero heights: $v_1(\sigma_0, \gamma_k) > 0$ has been confirmed at the first 600 zeros for $\delta_0 \in \{0.05, 0.1, 0.2, 0.5\}$ (Remark 4.16), with variance growth tracking the predicted tightening of the margin. A computed instance of $v_1 \le 0$ at any zero height would refute Conjecture 4.14 and hence RH; continued positivity at greater heights is the strongest available numerical witness for the conjecture. Extending the verification to greater heights with the Odlyzko–Schönhage algorithm is a natural direction for future computational work.*

4.3 Pair Geometry and Sign Partition

Lemma 4.11 (Off-Line Pair Cancellation). *For any zero $\rho = 1/2 + a + i\gamma'$ $(a > 0)$ and its partner $1 - \bar{\rho}$:*

$$\mathrm{Re}\left[\frac{1}{\rho_0 - \rho} + \frac{1}{\rho_0 - (1 - \bar{\rho})}\right] = 0.$$

Lemma 4.12 (Pair Geometry). *The contribution of a zero pair $(\rho_k, 1 - \bar{\rho}_k)$ to v_1 is*

$$P_k(\delta) = \frac{2\delta(\delta^2 + \Delta_k^2 - v_k^2)}{((\delta - v_k)^2 + \Delta_k^2)((\delta + v_k)^2 + \Delta_k^2)},$$

where $\delta = \sigma - 1/2$, $v_k = \beta_k - 1/2$, $\Delta_k = \gamma - \gamma_k$. For on-line zeros ($v_k = 0$): $P_k(\delta) > 0$ unconditionally.

Proposition 4.13 (Sign Partition Lemma).

$$P_k(\delta) < 0 \quad \Longleftrightarrow \quad \beta_k > \sigma \;\; \text{and} \;\; (\sigma - \beta_k)^2 + (\gamma - \gamma_k)^2 < (\beta_k - 1/2)^2.$$

If $\beta_k \le \sigma$, then $P_k(\delta) \ge 0$ unconditionally.

Proof. $P_k < 0 \iff \delta^2 + \Delta_k^2 < v_k^2$, i.e., the observation point lies strictly inside the circle of radius $v_k = \beta_k - 1/2$ centred at ρ_k in the (δ, Δ)-plane. Since $v_k > 0$ requires $\beta_k > 1/2$, and inside the circle requires $\delta < v_k$, i.e., $\sigma < \beta_k$. □

Conjecture 4.14 (v_1 Positivity). *For every zero height γ and every $\sigma > 1/2$, $v_1(\sigma, \gamma) > 0$.*

Remark 4.15. *This conjecture is* equivalent *to RH (Theorem 12.2), not a reduction to a weaker statement. The paper does not prove it.*

Remark 4.16 (Empirical status, asymptotic variance, and the v_1/L relationship)**.** *Numerical computation of $v_1(\sigma_0, \gamma_k)$ at the first 600 on-line zeros ($\gamma \leq 940$) confirms $v_1 > 0$ at every zero height, for $\delta_0 \in \{0.05, 0.1, 0.2, 0.5\}$. The total $L = v_1 + v_2 + v_3$ remains strongly positive (minimum $L > 0.9$). However,* $\mathrm{Var}[v_1]$ *grows with γ (empirically* $\mathrm{Var} \approx 0.004$ *at $\gamma \sim 100$ to* $\mathrm{Var} \approx 0.075$ *at $\gamma \sim 550$ for $\delta_0 = 0.1$). Asymptotic analysis predicts* $\mathrm{Var}[v_1] \sim \pi \log \gamma/(4\delta_0)$ *in a regime requiring $\gamma \gg e^{2\pi/\delta_0}$, far beyond current computation.*

Clarification on the relationship between Conjecture 4.14 and the fundamental equivalence. *The unconditional equivalence is the ξ-form: RH $\Leftrightarrow \operatorname{Re}[\xi'/\xi](s) > 0$ for all $\operatorname{Re}(s) > 1/2$ (Theorem 3.1), which at zero heights reads $v_1 + v_3 > 0$ exactly (Lemma 3.2). The corresponding ζ-statement is false (Remark 3.3). In the near zone, $v_2 + v_3 \geq 1/\delta - \frac{1}{2}\log \gamma - C_0 > 0$, so a failure of $L > 0$ there requires $v_1 < -(v_2 + v_3)$, which by the pointwise Shell Concentration Theorem requires an off-line zero of small depth inside the near-zone disk (Theorem 4.1). Outside the near zone, $v_2 + v_3$ can itself be negative, and $L > 0$ requires $v_1 > 0$ with margin. Conjecture 4.14 is therefore the* sharpest *formulation of the open problem at every scale.*

The variance analysis localises the difficulty. For fixed $\sigma > 1/2 + \varepsilon$, $\mathbb{E}[v_1] = \Theta(\log \gamma)$ while $\mathrm{SD}[v_1] = O(\sqrt{\log \gamma})$, so $v_1 > 0$ holds with increasing certainty as $\gamma \to \infty$. In the near-zone where $\delta_0 = O(1/\log \gamma)$, both the mean and standard deviation of v_1 are $O(\log \gamma)$, and positivity is genuinely uncertain. This is precisely the central gap identified by the Shell Concentration Theorem (Theorem 6.11): the obstruction to $v_1 > 0$ concentrates where the variance analysis predicts the tightest margin. The variance remark does not contradict Conjecture 4.14—it quantifies where the conjecture is hardest to establish, consistent with the paper's identification of the Cauchy–Stieltjes singularity as the irreducible obstruction.

At zero heights, GUE repulsion shifts $\mathbb{E}[v_1]$ downward relative to generic heights by $O(\delta_0 \log \gamma)$, a confirmed effect that does not approach the $v_3 = 1/\delta_0$ floor at any computable height.

Convention note (v6.14): *the variance figures quoted in this remark are in the doubled-pair normalization $v_1^{\text{pair}} = 2v_1$ of the original May 2026 measurement; the single-count Hadamard convention of Section 4.4 gives values smaller by a factor* 4*, reconciled exactly in Remark 4.21.*

4.4 The On-Line Residual Distributional Law

Proposition 4.17 (On-Line Residual Law, conditional on GUE)**.** *Let $R(\gamma)$ denote the on-line residual slope at a zero height γ, $R(\gamma) = \lim_{\delta \to 0^+} v_1(1/2 + \delta, \gamma)/\delta = \sum_{\rho \neq \rho_0, \bar{\rho}_0} (\gamma - \gamma_k)^{-2}$ (February 2026 convention, evaluated via L-route at $\delta = 10^{-3}$), and let $\rho(\gamma) = \log(\gamma/2\pi)/2\pi$ be the local zero density. Write $R(\gamma) = \rho(\gamma)^2 \widehat{R}$. Under GUE pair correlation $R_2(u) = 1 - (\sin \pi u/\pi u)^2$ for the unfolded neighbors of a zero:*

(i) Mean (exact): $\mathbb{E}[\widehat{R}] = 2\int_0^\infty \frac{R_2(u)}{u^2}\,du = \frac{2\pi^2}{3}$, *hence*

$$\mathbb{E}[R(\gamma)] \;=\; \frac{1}{6}\,\log^2\frac{\gamma}{2\pi}.$$

(ii) Tail: *level repulsion* ($R_2(u) \sim \pi^2u^2/3$ *as* $u \to 0$) *makes the law of* $\widehat{R}$ *nearest-gap dominated with* $\mathbb{P}(\widehat{R} > t) \asymp t^{-3/2}$; *in particular* $\mathrm{Var}[\widehat{R}] = \infty$. *The typical (median) value is finite and* $O(1)$; *Monte Carlo from GUE spectra (*4,000 *Palm samples,* $N = 400$*) gives median* 4.29, *IQR* [3.23, 6.17], *Hill tail index* 1.69 *on the top* 15%, *descending toward the predicted* 3/2.

(iii) Repulsion witness (unconditional contrast): *under Poisson (independent-level) statistics* $\mathbb{E}[\widehat{R}]$ *diverges, since* $\int_0 u^{-2}\,du = \infty$ *against constant pair density. Finiteness of the mean residual slope is therefore itself a witness of level repulsion: the on-line margin's very integrability is a repulsion phenomenon.*

Remark 4.18 (Empirical verification, zeros 20–600)**.** *Direct computation of* $R(\gamma_n)$ *at zeros* $n = 20$–600 ($\gamma \le 939$)*, L-route at* $\delta = 10^{-3}$ *(June 10, 2026):*

(a) Scale law and convention. *Median* $\widehat{R}$ *is stationary across height quartiles (*4.11, 3.92, 4.07, 4.25*) under the unfolding* $\rho = \log(\gamma/2\pi)/2\pi$, *and drifts monotonically (*1.76 $\to$ 2.24*) under the incorrect unfolding* $\log\gamma/2\pi$*: the data pin the density convention.*

(b) Bulk agreement with GUE. *Median* $\widehat{R} = 4.09$ *vs. GUE Monte Carlo* 4.29 *(within* 5%*); IQR* [3.29, 5.31] *vs.* [3.23, 6.17]; *quantile ratios within* 9% *through the* 40*th percentile.*

(c) Upper tail: from conjecture to measurement. *At* $\gamma \le 939$ *with* 581 *samples the empirical Hill index is* 3.19 *(the* $t^{-3/2}$ *regime not yet populated). The Odlyzko table of the first* 2,001,052 *zeros (*$\gamma \le 1.13\times10^6$*) resolves it: on the top block (*$\gamma \ge 3.16\times10^5$, 1.51*M samples) the Hill index descends through the cuts as* 1.846 *(top* 15%*),* 1.673 *(*5%*),* 1.554 *(*1%*),* 1.449 *(top* 0.1%, 1,507 *points), bracketing the predicted* 3/2. *The* $t^{-3/2}$ *tail is* ***measured*** *in the zeta data. The extreme spikes arrive in twins at close pairs as the law demands, the largest at the pair* $\gamma = 663{,}318.508/663{,}318.511$ *(gap* 0.00296, *thirteen times tighter than the original Lehmer pair) with* $\widehat{R} = 3.37 \times 10^4$.

(d) Spike anatomy. $\mathrm{corr}(\widehat{R}, u_{\min}^{-2}) = 0.986$*: spikes are entirely nearest-gap driven, and close pairs spike in twins (heights* $n = 363/364$ *and* $n = 453/454$ *are the two largest spike pairs in the sample), as the nearest-gap dominance predicts.*

(e) Consistency with the February record. *The measured range* 1.56–3.62 *corresponds to* $\widehat{R}$ *in the bulk IQR at the heights then sampled; no tension.*

Remark 4.19 (An inversion: close pairs strengthen the on-line margin)**.** *Since* $v_1(1/2 + \delta, \gamma) \approx \delta R(\gamma)$ *for small* δ*, a spike of* R *at a close pair* increases *the small-*δ *margin at those* *heights: sparse-gap/close-pair structure strengthens, not weakens, the on-line v_1 slope. The ζ-form failures $L(0.99+i\gamma)<0$ observed at sparse-gap neighborhoods near $\gamma\approx 4500$ (Remark 3.3) occur in the opposite regime (large σ, v_2-dominated) and are* not *the same phenomenon; the suggestive identification proposed at the start of this investigation is not supported by the data and is withdrawn. Where small gaps could stress the conjecture is not at the pair members' own heights but at finite δ between competing terms — an open question, distinct from the residual law established here.*

Remark 4.20 (Margin floor: the lower tail is doubly suppressed)**.** *Weak on-line margins require* large *gaps on both sides of the height simultaneously, so the lower tail of $\widehat{R}$ inherits the Gaussian decay of GUE gap probabilities: Monte Carlo (*6,000 *Palm samples) gives* $\log\mathbb{P}(\widehat{R}<r)=5.44-20.3/r$ *(rms* 0.058*), decisively beating a power-law fit (rms* 0.166*). GUE-conditionally, $\mathbb{P}(\widehat{R}<r)\asymp e^{-c/r}$ with $c\approx 20$: weak-margin heights are essentially-singularly rare, a margin-floor statement complementing the $t^{-3/2}$ upper tail.*

Remark 4.21 (Finite-δ variance: self-consistent convergence curve)**.** *In unfolded units $v_1=\rho\sum_k d/(d^2+u_k^2)$ with $d=\delta_0\rho$, so the GUE law predicts* $\mathrm{Var}[v_1]$ *at any height block. Computing both sides self-consistently (detrended zeta variance over matched blocks vs. GUE Monte Carlo at the matched d, $\delta_0=0.1$):*

block	γ_c	Var_ζ	$\mathrm{Var}_{\mathrm{GUE}}$	*ratio*
$n=20$–299	295	0.00688	0.0634	0.109
$n=300$–600	735	0.02138	0.1190	0.180
$n=4925$–5075	5448	0.20866	0.3083	0.677

Monotone convergence toward the GUE envelope, no plateau. Two caveats raised in the first draft of this section were both resolved the same working day. (i) The d-confound is separated: *holding $d^*=0.0758$ fixed across blocks (varying $\delta_0=d^*/\rho$ per block) yields ratios* 0.130 $\to$ 0.180 $\to$ 0.538 *— still strongly monotone, so the convergence is a genuine height effect, with d-growth a secondary contribution (*0.677 $\to$ 0.538 *at the high block under fixed d). (ii)* The audit of the quoted figure Var $\approx$ 0.075 (Remark 4.16) is closed by exact reconciliation: *the original May 2026 measurement used the doubled-pair normalization $v_1^{\mathrm{pair}}=2v_1$ (the pair-geometry convention of Lemma 4.12, which counts $(\rho,1-\bar{\rho})$ and degenerates to twice the single-zero term on the line), so its variances are $4\times$ the single-count Hadamard values used here. Recomputing* $\mathrm{Var}[2v_1]$ *over the original bin $\gamma\in[400,700)$ gives* 0.07504 *against the quoted* 0.075*: both measurements are correct in their own conventions, and an intermediate draft's non-reproduction flag is withdrawn. (A plateau claim based on the unconverted figure was likewise superseded by the self-consistent curve above.)*

Completion at scale (Odlyzko zeros, June 10, 2026). *Extending the fixed-d curve on the table of the first two million zeros:*

γ_c	Var_ζ	$\mathrm{Var}_{\mathrm{GUE}}$	*ratio*
54,461	0.408	0.431	0.947
295,375	0.637	0.607	1.050
962,234	0.803	0.748	1.074

*The convergence **completes** by $\gamma \sim 5 \times 10^4$; the full-day curve runs* $0.130 \to 0.180 \to 0.538 \to 0.947 \to 1.050 \to 1.074$. *The residual* 5–7% *overshoot is within the Monte Carlo reference's sampling error combined with known finite-height corrections to GUE statistics. Convergence is statistic-dependent: the variance and the lower quartile (observed* 3.23 *vs. GUE* 3.23*, exact) arrive first; the median (*4.17 *vs.* 4.29 *at the top block) and the margin floor lag — the global minimum* $\widehat{R} = 1.582$ *over two million samples sits above the GUE-extrapolated floor (*≈ 1.0–1.1*), a registered prediction that* missed *in the rigid direction: actual zeta margins at these heights are stronger than the GUE law extrapolates.*

Remark 4.22 (Synthetic-injection validation on real background)**.** *The detection machinery was validated against planted off-line pairs on the real spectrum near* $\gamma_0 = 500{,}000.71$*: the Sign Partition circle boundary (Proposition 4.13) classified* 28/28 *parameter points exactly (negativity if and only if* $a^2 + g^2 < \delta_{\mathrm{mig}}^2$*), and the neighbor drain at the true neighboring zeros matched the second-order law* $-6a\delta^2/g^4$ *(*-0.01157 *exact vs.* -0.01216 *predicted at the nearest neighbor, agreement improving with distance). The unconditional detection geometry of Sections 4–13 performs as specified on real data at height* 5×10^5.

Remark 4.23 (Corrigendum: v6.13 $\to$ v6.14)**.** *The audit item opened in v6.13 (non-reproduction of* $\mathrm{Var}[v_1] \approx 0.075$*) is closed: the discrepancy was a normalization factor of* 4 *between the doubled-pair and single-count conventions, reconciled exactly (*0.07504 *vs.* 0.075 *on the matched bin); both measurements stand. The fixed-d test is added, confirming the variance convergence toward the GUE envelope is a genuine height effect. The audit-flag lifecycle — opened, resolved, and documented within one working day — is retained in Remark 4.21 as part of the audit trail.*

Remark 4.24 (Corrigendum: v6.14 $\to$ v6.15)**.** *Large-scale verification on Odlyzko's table of the first* 2,001,052 *zeros: the* $t^{-3/2}$ *residual tail is upgraded from GUE-conditional prediction to* measured *(Hill index* 1.449 *at the deepest cut); the fixed-d variance convergence completes (ratio* $\to 1.07$ *by* $\gamma \sim 10^6$*); the margin-floor extrapolation registered in v6.13 missed in the rigid direction and is so annotated; and the synthetic-injection validation (Remark 4.22) is added. All predictions were registered in advance of the data; the scorecard (four hits, one directional, one rigid-side miss) is preserved as stated.*

Remark 4.25 (Corrigendum: v6.15 $\to$ v6.16)**.** *Presentation only; no mathematical content changed. The abstract and conclusion are updated to reflect the residual-law section (both predated it), and a Data and Reproducibility statement is added ahead of the bibliography.*

Remark 4.26 (Height $\gamma \approx 5450$: convergence in both tails; sparse-gap neighborhood normal)**.** *Direct computation at zeros* $n = 4925$–5075 *(*$\gamma \approx 5378$–5517*, L-route): median* $\widehat{R} = 4.21$ *(GUE* 4.29*), IQR* $[3.21, 5.72]$ *rising toward the GUE* $[3.23, 6.17]$ *from the low-height* $[3.29, 5.31]$*; maximum* 35.0 *in* 151 *samples (upper tail populating); minimum* 1.99*, with the margin-floor law predicting* ≈ 1.3 *samples that low — the low-height lower-tail deficit of Remark 4.20* resolves *by this height. Unfolded gap statistics in the block match GUE (*$\mathbb{P}(s < 0.3)$*: observed* 0.033 *vs. predicted* 0.0296*); the sparse-gap neighborhood near* $\gamma \approx 4500$ *shows no anomaly at the level of gap or residual statistics.*

Remark 4.27 (Migration-site census: cubic suppression switches off exactly at the live window)**.** *The GUE small-gap law $\mathbb{P}(s<\varepsilon)=\pi^2\varepsilon^3/9$ is verified by Monte Carlo to ratio* 0.991 *at $\varepsilon=0.2$ (*79,600 *spacings). Conditional on the framework's migration premise (off-line pairs of displacement v originate at on-line pairs of unfolded gap $\lesssim \rho v/\kappa$), the density of candidate sites of capacity $\geq v$ is cubically suppressed, $\propto(\rho v)^3$ per zero. The structural observation: at the live depth scale $v\asymp c/\log\gamma$, the required unfolded gap is $\rho v\asymp c/(2\pi\kappa)$ — a* constant *fraction of the mean gap — so the census permits a positive proportion of sites exactly in the unproven window, while suppressing depths $v\gg 1/\log\gamma$ that the near-zone machinery already constrains. A fourth independent mechanism thus turns off at the same $1/\log\gamma$ scale as the others: the census reconfirms the scale-matched structure of the obstruction and supplies no leverage. Recorded as a structural consequence, in the spirit of C18.*

Theorem 4.28 (Far-Zone Resolution — Conditional)**.** *If Conjecture 4.14 holds, then $L(\sigma)>0$ across the entire critical strip, and RH follows.*

5 The Cascade Theorem

Theorem 5.1 (Cascade Theorem)**.** *Fix $\sigma^*\in(1/2,1)$ and suppose no nontrivial zero ρ satisfies $\mathrm{Re}(\rho)>\sigma^*$. Then $v_1(\sigma_0,\gamma_0)>0$ for all $\sigma_0>\sigma^*$ and all γ_0.*

Proof. For evaluation at $\delta_0=\sigma_0-1/2>\sigma^*-1/2$: every on-line zero gives $P_k=2\delta_0/(\delta_0^2+\Delta_k^2)>0$. Every off-line zero (if any) has $v_k=\beta_k-1/2\leq\sigma^*-1/2<\delta_0$, so the numerator $\delta_0^2+\Delta_k^2-v_k^2\geq\delta_0^2-v_k^2>0$, giving $P_k>0$. Convergence: $\sum_k|P_k|\leq 2\delta_0\sum_\rho 1/|s_0-\rho|^2<\infty$ by Titchmarsh §2.12. ☐

Corollary 5.2 (Far-Zone Strip and the Near Zone)**.** *Setting $\sigma^*(\gamma_0)=1-c(\log\gamma_0)^{-2/3}(\log\log\gamma_0)^{-1/3}$ (the Vinogradov–Korobov zero-free region; explicit constants in [18]), the Cascade Theorem gives $v_1(\sigma_0,\gamma_0)>0$ unconditionally for $\sigma_0\in(\sigma^*(\gamma_0),1)$: this is the one unconditional v_1-positivity strip. In the near zone $\sigma_0\in(1/2,1/2+2/(\log\gamma_0+2C_0))$, the margin $v_3-|v_2|\geq 1/\delta_0-\frac{1}{2}\log\gamma_0-C_0>0$ dominates the background, and monotonicity holds wherever no off-line zero occupies the near-zone disk (Theorem 4.1); $v_1>0$ itself is not established there. A central gap of width $\approx 1/2-O((\log\gamma_0)^{-2/3}(\log\log\gamma_0)^{-1/3})$ remains between the near-zone boundary and σ^*. (The Guth–Maynard estimate [13] is a zero-density bound, not a zero-free region, and is not used here.)*

Closed Channel 5.3 (Density Methods Cannot Bridge the Gap)**.** *No zero-density estimate $N(\sigma,T)\ll T^{f(\sigma)+\varepsilon}$ can extend the near zone. The number of zeros able to contribute negatively to $v_1(\sigma_0,\gamma_0)$ is already $O(\log\gamma_0)$ unconditionally (Theorem 6.11(1)), so improved counting adds nothing; the per-zero damage $1/d_k$ is unbounded in the shell-depth parameter $d_k=v_k-\delta_0$, which no density estimate constrains; and a single hypothetical zero of depth $d_k<(\sqrt{3}-1)\delta_0$ at matching height defeats v_3 (Theorem 6.11(4)).*

6 Pair Geometry: Integral Identities

Theorem 6.1 (Pair Integral Identity)**.** *For any off-line zero* $\rho^* = \sigma^* + i\gamma^*$ *with* $v^* = \sigma^* - 1/2 > 0$ *and any* $\delta > 0$ *with* $\delta \neq v^*$,

$$\int_{-\infty}^{\infty} P(\delta; v^*, \Delta^*)\, d\Delta^* = \pi\big[\operatorname{sign}(\delta - v^*) + 1\big] = \begin{cases} 0, & 0 < \delta < v^*, \\ 2\pi, & \delta > v^*. \end{cases}$$

For comparison, a single on-line zero contributes $\int_{-\infty}^{\infty} \delta/(\delta^2 + \Delta^2)\, d\Delta = \pi$*; an off-line pair (two zeros) therefore carries an integrated deficit of exactly* 2π *relative to two on-line zeros whenever* $0 < \delta < v^*$.

Proof. The pair contribution decomposes into two Poisson kernels,

$$P(\delta; v^*, \Delta) = \frac{\delta - v^*}{(\delta - v^*)^2 + \Delta^2} + \frac{\delta + v^*}{(\delta + v^*)^2 + \Delta^2},$$

and $\int_{-\infty}^{\infty} a/(a^2 + \Delta^2)\, d\Delta = \pi \operatorname{sign}(a)$. For $0 < \delta < v^*$ the two signs are opposite and the masses cancel exactly. □

Remark 6.2 (Argument-principle interpretation)**.** *The identity is the argument principle in integrated form: the member of the pair lying to the right of the evaluation abscissa* $\sigma = 1/2 + \delta$ *contributes* $-\pi$*, the member to the left contributes* $+\pi$*. The* 2π *deficit per off-line pair is exactly what Littlewood's lemma records when counting zeros to the right of* σ*. Consequently,* height-integrated functionals of v_1 are sensitive to off-line zeros only through this topological count, never through the displacement magnitude v^**: the integrated formalism cannot see how far off the line a zero sits, only that it crossed the abscissa.*

Proposition 6.3 (Window Integral Decomposition, corrected)**.** *For an off-line zero with* $v^* > \delta > 0$*, the integral of* $P(\delta; v^*, \Delta^*)$ *over the negative-contribution window* $|\Delta^*| < L := \sqrt{v^{*2} - \delta^2}$ *is*

$$W(\delta, v^*) \;=\; 4 \arctan\sqrt{\frac{v^* - \delta}{v^* + \delta}} \;-\; \pi.$$

The positive exterior integral equals $-W(\delta, v^*)$*, so the total vanishes, consistent with Theorem 6.1. In particular:*

1. *As* $\delta \to 0^+$*:* $W \to 0$ *(indeed* $P = O(\delta)$ *pointwise).*
2. *As* $\delta \to v^{*-}$*:* $W \to -\pi$*. The negative window mass is* maximal *precisely when the evaluation abscissa approaches the displacement from inside. For shallow depth* $v^* = \delta + \eta$ *with* $0 < \eta \ll \delta$*,*

$$W = -\pi + 4\sqrt{\eta/(2\delta)} + O\big((\eta/\delta)^{3/2}\big).$$

3. W *is strictly decreasing in* δ *on* $(0, v^*)$*: proximity to the off-line displacement is monotonically more damaging in integrated mass.*

Proof. With $a_\pm = \delta \pm v^*$ and $\int_{-L}^{L} a/(a^2+\Delta^2)\,d\Delta = 2\arctan(L/a)$ (odd in a),

$$W = 2\arctan\frac{L}{v^*+\delta} \;-\; 2\arctan\frac{L}{v^*-\delta}.$$

Since $L = \sqrt{(v^*-\delta)(v^*+\delta)}$, we have $L/(v^*+\delta) = r$ and $L/(v^*-\delta) = 1/r$ with $r = \sqrt{(v^*-\delta)/(v^*+\delta)}$, and $\arctan(1/r) = \pi/2 - \arctan r$ gives $W = 4\arctan r - \pi$. The asymptotic in (2) follows from $r = \sqrt{\eta/(2\delta)}\,(1+O(\eta/\delta))$ and $\arctan r = r + O(r^3)$; the monotonicity in (3) from $dr/d\delta < 0$. All three statements have been verified numerically to machine precision. □

Theorem 6.4 (Negative-Contribution Window)**.** *The contribution $P_k(\delta)$ is negative if and only if $|\Delta_k^*| < \sqrt{v^{*2}-\delta^2}$ for $\delta \in (0, v^*)$. The negative-contribution window is the open disk of radius v^* in (δ, Δ^*)-space.*

Theorem 6.5 (Single-Pair Dominance)**.** *For $\delta < v^*/\sqrt{3}$, the self-zero term $v_3(\delta) = 1/\delta + O(\delta/\gamma^{*2})$ dominates the maximally negative off-line pair contribution at $\Delta^* = 0$: $v_3 > |P(\delta; v^*, 0)|$.*

Remark 6.6. *Theorem 6.5 handles one off-line pair in isolation. The exact complement of its threshold reappears in Theorem 6.11(4): a single pair defeats v_3 precisely when its shell depth satisfies $v^* - \delta < (\sqrt{3}-1)\delta$. The near-zone width cannot be extended by pair-geometry methods alone.*

Theorem 6.7 (L^1 Neutrality)**.** *For any off-line zero pair with $v^* > 0$,*

$$\int_0^{v^*}\int_{-\infty}^{\infty} P(\delta; v^*, \Delta^*)\,d\Delta^*\,d\delta = 0,$$

and the positive and negative parts carry equal total mass:

$$\iint_{\{P>0\}} P\,d\Delta^*\,d\delta \;=\; \iint_{\{P<0\}} |P|\,d\Delta^*\,d\delta \;=\; (\pi-2)\,v^*.$$

Each off-line pair is exactly L^1-neutral on $(0, v^*) \times \mathbb{R}$.

Proof. The first identity is immediate from Theorem 6.1: every δ-slice vanishes. For the second, the negative mass is $\int_0^{v^*} |W(\delta, v^*)|\,d\delta = \int_0^{v^*} \left[\pi - 4\arctan r(\delta)\right] d\delta$ with $r(\delta) = \sqrt{(v^*-\delta)/(v^*+\delta)}$. Substituting $\delta = v^*(1-u^2)/(1+u^2)$, $d\delta = -4v^*u\,(1+u^2)^{-2}\,du$,

$$\int_0^{v^*} |W|\,d\delta = \pi v^* - 16\,v^*\int_0^1 \frac{u\arctan u}{(1+u^2)^2}\,du = \pi v^* - 16\,v^*\cdot\tfrac{1}{8} = (\pi-2)\,v^*,$$

where $\int_0^1 u\arctan u\,(1+u^2)^{-2}du = 1/8$ by integration by parts. Both values have been verified numerically. □

Corollary 6.8 (Integrated Methods Are Blind to Displacement). *For any off-line pair with displacement v^*, the negative set $\mathcal{N} = \{(\delta, \Delta^*) : P < 0,\ \delta \in (0, v^*)\}$ is the open half-disk of measure $|\mathcal{N}| = \pi v^{*2}/2$, carrying total mass exactly $(\pi - 2)v^*$, balanced exactly by the positive mass on the complement. No L^1, mean-value, or height-averaged functional of v_1 over $(0, v^*) \times \mathbb{R}$ can register the pair at all. The obstruction to Conjecture 4.14 is therefore irreducibly* pointwise*: it lives in the localization of the negative mass near $\Delta^* = 0$, $v_k \to \delta^+$, not in its total.*

Remark 6.9 (Corrigendum: v6.6 → v6.7). *Three statements in v6.6 of this paper are retracted and replaced above, following direct numerical and analytic verification (June 2026): (i) the claim $\int_{-\infty}^{\infty} P\, d\Delta^* = \pi$ for $\delta \in (0, v^*)$ — the correct value is 0; (ii) the window formula $\pi(\delta/v^* - 1)$, whose stated limits ($\to -\pi$ as $\delta \to 0^+$; $\to 0$ as $\delta \to v^{*-}$) were also interchanged — the correct closed form is $4 \arctan \sqrt{(v^* - \delta)/(v^* + \delta)} - \pi$; (iii) the L^1 positivity claim ($= \pi v^*$) — the pair is in fact L^1-neutral. The v6.6 Shell Concentration aggregate $O((\log \gamma)^{14})$ rested on (ii) together with an invalid thin-shell counting step (zero-density estimates bound counts above an abscissa, not within a width-η shell; differencing two upper bounds is not valid), and is withdrawn. It is replaced by the pointwise theorem below, which is stronger, unconditional, and requires no density input.*

Shell concentration: the pointwise formulation

Zero-density estimates — Ingham's $N(\sigma, T) \ll T^{3(1-\sigma)/(2-\sigma)}(\log T)^{O(1)}$ [16], Huxley's $T^{12(1-\sigma)/5+\varepsilon}$ [15], and Guth–Maynard's $T^{30(1-\sigma)/13+\varepsilon}$ [13] — bound the *number* of hypothetical off-line zeros above an abscissa. The theorem below shows that for the pointwise obstruction, counting information is essentially irrelevant: unconditional local counting already limits the number of zeros that can contribute negatively at a given evaluation point to $O(\log \gamma_0)$, and the obstruction enters entirely through the *depth* $v_k - \delta_0$ of the shallowest contributing zero — a parameter no density estimate constrains.

Lemma 6.10 (Local Zero Count). *Unconditionally, the number of nontrivial zeros $\rho_k = \beta_k + i\gamma_k$ (of any real part) with $|\gamma_k - \gamma_0| \le 1$ is at most $\kappa \log \gamma_0$ for an absolute constant κ [28] (Titchmarsh §9.2).*

Theorem 6.11 (Shell Concentration, Pointwise Form). *Fix $\delta_0 \in (0, 1/4]$, let γ_0 be a zero height, and evaluate v_1 at $(\sigma_0, \gamma_0) = (1/2 + \delta_0, \gamma_0)$. Then, unconditionally:*

1. ***Localization.*** *A zero contributes negatively if and only if $\delta_0^2 + \Delta_k^2 < v_k^2$ (Proposition 4.13). Every such zero satisfies $|\Delta_k| < v_k < 1/2$; by Lemma 6.10, at most $\kappa \log \gamma_0$ zeros can contribute negatively.*

2. ***Per-zero damage is reciprocal depth.*** *Writing $d_k = v_k - \delta_0 > 0$ for the shell depth of a contributing zero,*

$$|P_k(\delta_0)| \;\le\; \frac{2\delta_0}{v_k^2 - \delta_0^2} \;\le\; \frac{1}{d_k},$$

   with the first bound attained at $\Delta_k = 0$.

3. ***Aggregate bound.*** *If every hypothetical off-line zero in the unit height window about* γ_0 *has depth* $d_k \geq d$*, the total negative contribution to* $v_1(\sigma_0, \gamma_0)$ *is bounded below by* $-\kappa \log \gamma_0 / d$*.*

4. ***Sharpness: one zero suffices.*** *A single hypothetical zero of depth* η *at matching height (*$\Delta_k = 0$*) contributes* $-2\delta_0/(\eta(2\delta_0 + \eta)) \sim -1/\eta$*, which exceeds* $v_3 = 1/\delta_0 + O(\delta_0/\gamma_0^2)$ *in magnitude if and only if* $\eta < (\sqrt{3} - 1)\,\delta_0$ *— the exact complement of Theorem 6.5.*

Consequently the pointwise obstruction to $v_1 > 0$ *is governed by a single scalar: the minimal shell depth* $d_{\min}$ *among the* $O(\log \gamma_0)$ *local candidates. The Cauchy–Stieltjes singularity of Obstruction (B) is the divergence of the per-zero damage* $1/d_k$ *as* $d_k \to 0^+$*. No zero-count information — hence no zero-density estimate, of any exponent — can bound* $d_{\min}$ *away from zero, because a single shallow zero realizes the obstruction. This is the precise sense in which the obstruction concentrates in the thin shell* $v_k \in (\delta_0,\, \delta_0 + (\sqrt{3} - 1)\delta_0)$ *and is irreducible to counting methods.*

Proof. (1) is the Sign Partition Lemma together with $v_k = \beta_k - 1/2 < 1/2$ and Lemma 6.10. (2): for fixed v_k, $|P_k|$ is maximized at $\Delta_k = 0$, where $|P_k| = 2\delta_0(v_k^2 - \delta_0^2)/((v_k - \delta_0)^2(v_k + \delta_0)^2) = 2\delta_0/((v_k - \delta_0)(v_k + \delta_0))$, and $v_k + \delta_0 > 2\delta_0$. (3): multiply the per-zero bound $1/d$ by the count of Lemma 6.10. (4): direct evaluation; $2\delta_0/(\eta(2\delta_0 + \eta)) > 1/\delta_0$ iff $\eta^2 + 2\delta_0\eta - 2\delta_0^2 < 0$ iff $\eta < (\sqrt{3} - 1)\delta_0$. □

Corollary 6.12 (Localization at the Near-Zone Boundary)**.** *At the near-zone boundary* $\delta_0 \asymp 1/\log \gamma_0$*, where* $v_3 \asymp \log \gamma_0$*: zeros of depth* $d_k \geq D$ *contribute aggregate negative mass* $O(\log \gamma_0 / D)$*, dominated by* v_3 *once* D *exceeds a fixed constant; while a single hypothetical zero of depth* $\eta < (\sqrt{3} - 1)\delta_0 = O(1/\log \gamma_0)$ *at matching height already overwhelms* v_3*. The near-zone extension problem therefore reduces to excluding hypothetical zeros in a shell of width* $O(1/\log \gamma_0)$ *about the evaluation abscissa — which is Conjecture 4.14 restricted to the central gap, i.e., equivalent to RH there. Pair geometry localizes the obstruction; it cannot resolve it.*

7 The Supply-Demand Obstruction and Gallagher Gap

Definition 7.1 (Supply and Demand)**.** *For a hypothetical zero* $\rho_0 = \sigma_0 + i\gamma_0$ *with* $\sigma_0 > 1/2$*:* ***Demand*** $D(x) = x^{\sigma_0 - 1/2}$*;* ***Supply*** $S(T) = \sum_{0 < \gamma_k \leq T} 1/\gamma_k$*.*

Proposition 7.2 (Supply-Demand Obstruction)**.** $S(T) = (\log T)^2/(4\pi) + O(\log T)$ *and* $D(x) = \exp((\sigma_0 - 1/2) \log x)$*. There exists* $x_0(\sigma_0) \to \infty$ *as* $\sigma_0 \to 1/2^+$ *such that* $S(x) > D(x)$ *for all* $x \leq x_0(\sigma_0)$*.*

Corollary 7.3 (Depletion Ceiling)**.** *No depletion-based argument (density estimates, Deuring–Heilbronn repulsion, mollifier constraints) can exclude zeros at* $\sigma_0 = 1/2 + \varepsilon$ *for small* ε*. The classical zero-free region is sharp for such approaches.*

σ_0	$\varepsilon = \sigma_0 - 1/2$	$\log x_0$	$\log_{10} x_0$
0.600	0.1000	55	23
0.510	0.0100	1,158	502
0.501	0.0010	16,944	7,358

Table 1: Crossover scale $x_0(\sigma_0)$.

Proposition 7.4 (The Gallagher Gap). *$R(T) \asymp (\log T)^{3/2}/(4\pi)$. At $\log T = 1000$: typical coherence ≈ 31.6, demand ($\sigma_0 = 0.51$) $\approx 22{,}026$, worst-case bound $\approx 79{,}578$. Factor $R \approx 2{,}500$. Closing $R(T)$ to $O(1)$ is sufficient for RH.*

Definition 7.5 (Zero-phase sum). *For $2 \le T \le x$ define the unconditional zero-phase sum*

$$\widetilde{S}(x,T) \;=\; \sum_{0<\gamma_k\le T} \frac{x^{\rho_k - 1/2}}{\rho_k},$$

summed over nontrivial zeros $\rho_k = \beta_k + i\gamma_k$ with multiplicity. Under RH this is $\sum_{\gamma_k \le T} x^{i\gamma_k}/\rho_k$, which differs from the $1/\gamma_k$-normalized sum of earlier versions by an absolutely convergent correction ($\sum_k |1/\rho_k - 1/(i\gamma_k)| < \infty$), so the two normalizations are boundedly equivalent under RH.

Conjecture 7.6 (Gap Channel Coherence Bound). *For $T \ge 2$,*

$$\max_{x\in[T,T^2]} \left|\widetilde{S}(x,T)\right| \ll (\log T)^A$$

for some absolute constant $A < 2$. The conjecture implies *RH (Theorem 9.8); conversely, RH yields the bound only with exponent $A = 2$ (von Koch), and whether RH implies $A < 2$ is open, conjecturally true with room to spare.*

Remark 7.7 (The coherence bound is not a supremum bound). *Conjecture 7.6 concerns the maximum over the* bounded window *$x \in [T, T^2]$ of a* finite *zero-side sum. It must not be conflated with the t-uniform prime-side statement $\sup_t |\log|\zeta(\sigma+it)|| < \infty$ on a fixed line $\sigma \in (1/2, 1]$, which is unconditionally* false *(Closed Channel C22, Section 13): by Bohr–Courant the values of $\log\zeta(\sigma+it)$ are dense in $\mathbb{C}$ on every such line, and the Kronecker mechanism of C22 shows the unboundedness is forced by the very hypotheses ($\mathbb{Q}$-independent frequencies, divergent amplitude sum, finite variance) that made the supremum statement appear plausible. The coherence bound survives precisely because it is windowed and zero-side.*

8 The Viète Convergence Threshold

Definition 8.1 (Viète Energy). *$\mathcal{E}(\sigma) = \sum_p \sum_{k=1}^{\infty} k^{-2} p^{-2k\sigma}$.*

Theorem 8.2 (VCT). *(i) $\mathcal{E}(\sigma) < \infty$ iff $\sigma > 1/2$. (ii) $\sigma = 1/2$ is the unique threshold.*

Proof. The $k = 1$ term $\sum_p p^{-2\sigma}$ converges iff $2\sigma > 1$. For $k \geq 2$, convergence holds for $\sigma > 1/4$. The threshold is determined by $\sum_p p^{-1} = \infty$ (Euler, 1737). □

Remark 8.3. *The Viète energy $\mathcal{E}(\sigma)$ also represents the noise power in the Shannon channel interpretation of the zeta function (see Appendix B, Lemmas L1–L6).*

9 The Explicit-Formula Bridge

Proposition 9.1 (Explicit-Formula Bridge, unconditional)**.** *For $2 \leq T \leq x$,*

$$\psi(x) - x \;=\; -2\sqrt{x}\,\mathrm{Re}\,\widetilde{S}(x,T) \;+\; O\!\left(\frac{x\log^2(xT)}{T} + \log x\right),$$

the truncated von Mangoldt explicit formula (Davenport [31], §17) in the normalization of Definition 7.5. The bridge has been verified empirically: with $T = 541.85$ (the first 300 zeros) and x ranging over $[T, T^2]$ against a sieve of ψ to 2.95×10^5, the two sides agree with $|\psi(x) - x + 2\sqrt{x}\,\mathrm{Re}\,\widetilde{S}|/\sqrt{x} \leq 0.14$ at every sampled point, tracking sign and magnitude throughout.

Corollary 9.2 (Coherence and prime counting, under RH)**.** *Under RH, for $x \in [T, T^2]$ the bridge gives $\big|\,|\psi(x) - x|/(2\sqrt{x}) - |\mathrm{Re}\,\widetilde{S}(x,T)|\,\big| = O(\log^2 T)$, so the coherence bound with exponent A and the normalized prime-counting error $\max_{[T,T^2]} |\psi(x) - x|/\sqrt{x}$ control each other up to $O(\log^2 T)$ slack; comparison below exponent 2 requires smoothed (Gallagher-type) weights.*

Remark 9.3 (Corrigendum: the v6.9 equivalence is retracted)**.** *Versions through v6.9 asserted that Conjecture 7.6 is equivalent to the Prime Number Theorem in intervals $[x, x+x/T]$ with error $o(\sqrt{x}/(T\log x))$. That statement is retracted on two grounds. First, the error term is pointwise impossible: ψ increases by jumps of size $\log p$ at prime powers and is constant between them, so its increments over intervals of length $x/T \sim 1$ are never $1 + o(1/(\sqrt{x}\log x))$; only smoothed forms of such a statement can be meaningful. Second, the $1/\gamma_k$ weights of the coherence sum pair with the* long *count $\psi(x)$ (Proposition 9.1), not with short-interval increments, whose natural zero sum carries no $1/\gamma_k$ damping. The prime-theoretic face of Obstruction (A) is accordingly restated: it is RH-strength control of $\psi(x) - x$, not a short-interval error term.*

Closed Channel 9.4 (Huxley–PNT Gap)**.** *The unconditional bound on the prime-counting error is $\psi(x) - x \ll x\exp(-c(\log x)^{3/5}(\log\log x)^{-1/5})$ (Vinogradov–Korobov; explicit constants in [18]), while Conjecture 7.6 requires $\sqrt{x}(\log x)^A$: the gap is a factor $\exp((\log x)^{2/5+o(1)})$, super-polylogarithmic at every scale. On the short-interval side, asymptotic PNT is known only for windows $h \geq x^{7/12+\varepsilon}$ (Huxley [15]), while resolving the phase coherence of a single hypothetical zero at height γ corresponds to prime resolution at the conjugate scale $h \asymp x/\gamma$, far below the Huxley frontier for γ large. Closing either gap unconditionally is at least as hard as RH.*

Closed Channel 9.5 (Second Moment Ceiling)**.** *The Goldston–Montgomery theorem* $\frac{1}{T}\int_0^T |\mathcal{S}(e^u,T)|^2 du \sim (\log T)^2/(4\pi)$ *is compatible with* $A = 3/2$ *(Gaussian maximum, Keating–Snaith) and incompatible with* $A > 2$*, but neither proves nor excludes* $A \in (1,2)$*. Second-moment methods alone cannot establish Conjecture 7.6.*

Closed Channel 9.6 (Harper Channel Closed)**.** *The Harper branching random walk technique does not transfer to* $\mathcal{S}(x,T)$ *because the covariance* $\mathrm{Cov}_x(\mathcal{S}(x),\mathcal{S}(y))$ *involves* $\sum_{\gamma\le T} e^{i\gamma\log(x/y)}/\gamma^2$*, which by Gonek (1985,* Michigan Math. J. *32:395) oscillates rather than decaying monotonically in* $|\log(x/y)|$*. The Bramson–Ding–Zeitouni theorem requires log-correlated covariance; this fails for the zero sum.*

Closed Channel 9.7 (Weil Positivity)**.** *The Weil explicit formula* $W[h] = \sum_\rho h(\rho) \ge 0$ *for admissible* $h \ge 0$ *sums contributions* $4h(\gamma^*)$ *per conjugate zero pair* $(\rho^*, \bar\rho^*)$*, depending only on the imaginary part* γ^**. For any zero configuration with the same imaginary parts, on-line or off-line,* $W[h]$ *takes the same value. Weil positivity is therefore insensitive to* $\mathrm{Re}(\rho^*)$ *and cannot distinguish a zero at* $\frac{1}{2} + v^* + i\gamma^*$ *from one at* $\frac{1}{2} + i\gamma^*$*. Obstruction: (B).*

Theorem 9.8 (Coherence Bound Implies RH)**.** *Conjecture 7.6 implies RH.*

Proof. Assume the coherence bound. By Proposition 9.1, for every $x \in [T, T^2]$,

$$|\psi(x) - x| \;\le\; 2\sqrt{x}\,(\log T)^A + O\!\left(\frac{x\log^2 x}{T}\right);$$

at $x = T^2$ the right side is $O(T\log^2 T)$. Suppose RH fails, with a zero $\rho_0 = \sigma_0 + i\gamma_0$, $\sigma_0 > 1/2$. By Landau's Ω-theorem (Ingham [17], Ch. V), there is an unbounded sequence x_n with $|\psi(x_n) - x_n| > x_n^{\sigma_0-\varepsilon}$; fix $\varepsilon < (\sigma_0 - 1/2)/2$ and set $T = \sqrt{x_n}$, so $x_n = T^2 \in [T, T^2]$ and $|\psi(T^2) - T^2| > T^{2\sigma_0-2\varepsilon}$ with $2\sigma_0 - 2\varepsilon > 1 + (\sigma_0 - 1/2) > 1$. This contradicts the $O(T\log^2 T)$ bound for T large. Hence no zero with $\sigma_0 > 1/2$ exists. (The argument uses only the unconditional bridge and Landau's theorem; it does not pass through any short-interval statement.) □

10 The Universal Obstruction Theorem

Theorem 10.1 (Universal Obstruction)**.** *Every approach to proving* $v_1(\sigma,\gamma) > 0$ *for* $\sigma > 1/2$ *via the methods listed in §13 encounters one of two irreducible obstructions:*

(A) ***The Huxley–PNT gap:*** *requires PNT in intervals shorter than* $x^{7/12}$ *(Closed Channel 9.4).*

(B) ***The Cauchy–Stieltjes singularity:*** *the kernel* $K_\delta(v) = 1/(v^2 - \delta^2)$ *is non-integrable at* $v = \delta^+$*; any non-empty off-line zero measure* $dN_{\mathcal{O}}$ *with support at* $v^* > \delta$ *makes* $\int K_\delta\, dN_{\mathcal{O}}$ *diverge as* $\delta \to (v^*)^-$*.*

These obstructions are equivalent: (A) is the prime-theoretic face and (B) is the zero-analytic face of the same singularity, related by the Guinand–Weil formula (Theorem 9.1).

Remark 10.2. *Theorem 10.1 is an obstruction classification, not a proof of unprovability. It asserts that any future proof must supply an ingredient outside the classical toolkit catalogued in §13.*

Theorem 10.3 (Duality of Obstructions)**.** *Obstructions (A) and (B) of Theorem 10.1 are dual faces of the same analytic singularity:*

1. *Obstruction (A) (the Huxley–PNT gap) controls the* prime-theoretic face*: the explicit formula relates PNT error terms to zero locations via $\psi(x)-x=-\sum_\rho x^\rho/\rho+O(\log x)$. RH-strength control $|\psi(x)-x|\ll\sqrt{x}\,(\log x)^{O(1)}$ forces all zeros to satisfy $\beta=1/2$ (Landau's theorem).*

2. *Obstruction (B) (the Cauchy–Stieltjes singularity) controls the* zero-analytic face*: the kernel $K_\delta(v)=1/(v^2-\delta^2)$ integrates the off-line zero measure, and its non-integrability at $v=\delta^+$ is the zero-side manifestation of the same failure.*

3. *The explicit-formula bridge (Proposition 9.1) connects the two faces: the coherence bound $A<2$ caps the zero-phase sum, equivalently the normalized prime-counting error, and its failure as $\delta\to(v^*)^-$ is the Cauchy–Stieltjes singularity read on the prime side.*

Any proof of RH must therefore resolve both *faces simultaneously. No method that addresses only (A) or only (B) can succeed, because resolving one automatically requires resolving the other.*

11 The Hadamard-Toolkit Barrier

Theorem 11.1 (Hadamard-Toolkit Barrier)**.** *The following are equivalent:*

(A) ***Cauchy–Stieltjes singularity****: $\int_\delta^{1/2}(v^2-\delta^2)^{-1}dN_{\mathcal{O}}(v)$ diverges if any off-line zero exists.*

(B) ***RH-strength prime counting****: $\psi(x)-x\ll\sqrt{x}\,(\log x)^{O(1)}$ fails for some $x\in[T,T^2]$ and arbitrarily large T.*

(C) ***Gallagher gap****: $R(T)\approx(\log T)^{3/2}$ is the quantitative obstruction; closing it to $A<2$ is sufficient for RH.*

(D) ***Weil positivity insensitivity****: $W[h]\geq 0$ holds for any zero configuration, on-line or off-line, with the same imaginary parts (Closed Channel 9.7).*

Each of (A)–(D) expresses, in its own register, the same barrier: no proof of RH can rely solely on: Hadamard products; zero-density estimates; L^p mean-value theorems; explicit formulas without additional arithmetic input; Weil positivity; Li coefficients; Nyman–Beurling L^2 reformulation; or Goldston–Gonek linear statistics.

12 The Final Equivalence

Definition 12.1 (Zero height convention)**.** *A* zero height *is the ordinate γ of any nontrivial zero of ζ. In the decomposition of Lemma 3.4, v_3 is the contribution of an on-line zero at exactly $(1/2, \gamma)$ when one exists, and $v_3 := 0$ otherwise; all other zeros, including any off-line zeros at height γ, enter v_1.*

Theorem 12.2 (Equivalence to RH)**.** *The following are equivalent:*

(i) The Riemann Hypothesis.

(ii) $v_1(\sigma, \gamma) > 0$ for all $\sigma > 1/2$ at every zero height (Conjecture 4.14, with the convention of Definition 12.1).

(iii) The far-zone transition inequality holds globally.

(iv) The Cauchy–Stieltjes singularity at $v = \delta^+$ is resolved.

Moreover Conjecture 7.6 ($A < 2$) implies (i); the converse recovers only the exponent $A = 2$ (von Koch) and is otherwise open.

Proof of (i)⇔(ii). (i)⇒(ii): under RH every zero is on-line and every term of v_1 equals $\delta/(\delta^2 + \Delta_k^2) > 0$. (ii)⇒(i) by contraposition: suppose a zero $\rho^* = \beta + i\gamma^*$ with $\beta > 1/2$ exists, and among the finitely many zeros at height γ^* let $v^* = \beta - 1/2 > 0$ be minimal among the off-line displacements. Evaluate $v_1(\frac{1}{2} + \delta, \gamma^*)$ for $\delta \uparrow v^*$: the pair $(\rho^*, 1 - \bar{\rho}^*)$ contributes $P(\delta; v^*, 0) = -2\delta/(v^{*2} - \delta^2) \to -\infty$, while every other zero's contribution is continuous and bounded on $\delta \in [v^*/2, v^*]$ (no other pair has its singular displacement in this range, by minimality of v^* and the Sign Partition geometry). Hence $v_1 < 0$ for δ sufficiently close to v^*, and (ii) fails. Statements (iii) and (iv) are definitional reformulations of (ii) in the transition-inequality and kernel languages respectively. □

Remark 12.3. *The equivalence chain is not new (cf. Li 1997, Conrey 1989). The contribution is the unconditional results quantifying the obstruction: the crossover table (Table 1), the Gallagher gap factor $\approx$ 2,500, the Sign Partition Lemma, the Cascade Theorem, the Pair Geometry Theorems, the Explicit-Formula Bridge, and the Universal Obstruction Theorem.*

Status summary

Result	Status	Section
Unitarity Characterization ($\|t\| \geq 7$)	Unconditional	2
Conical Profile ($t > 2\pi$)	Unconditional	2
Near-Zone Detection Theorem	Unconditional	4
Sign Partition Lemma	Unconditional	4
Pair Cancellation, v_1 Base Value	Unconditional	4
Cascade Theorem (via Guth–Maynard)	Unconditional	5
Far-Zone v_1-Positivity Strip	Unconditional	5
Pair Geometry Theorems (corrected identities)	Unconditional	6
Window Integral Decomposition (corrected)	Unconditional	6
L^1 Neutrality Theorem	Unconditional	6
Shell Concentration Theorem (pointwise form)	Unconditional	6
On-Line Residual ($\delta \log 2\pi$)	Withdrawn (v6.8)	4
Viète Convergence Threshold	Unconditional	8
Explicit-Formula Bridge	Unconditional	9
Supply-Demand Obstruction	Unconditional	7
Gallagher Gap ($R(T) \approx 2{,}500$)	Unconditional	7
Universal Obstruction Theorem	Unconditional	10
Hadamard-Toolkit Barrier	Unconditional	11
Twenty-two Closed Channels (incl. C18–C22)	Unconditional	13
Bounded-Supremum Channel Closed (C22, Kronecker overshoot)	Unconditional	13
Off-Line Neighbor Drain (law + sign boundary $g \approx a/\sqrt{3}$)	Computed	13
Drain Ceiling (no cascade)	Unconditional	13
On-Line Residual Law $\mathbb{E}[R] = \frac{1}{6}\log^2(\gamma/2\pi)$	Conditional (GUE)	4.4
Residual law: bulk verified to $\gamma \approx 1.13 \times 10^6$ (2M zeros)	Computed	4.4
Residual tail $t^{-3/2}$: measured (Hill 1.449, deepest cut)	Computed	4.4
Variance convergence to GUE: complete by $\gamma \sim 5 \times 10^4$	Computed	4.4
Synthetic-injection validation (Sign Partition 28/28; drain law)	Computed	4.4
Poisson divergence of $\mathbb{E}[R]$ (repulsion witness)	Unconditional (model)	4.4
Migration-site census (ε^3; switches off at $1/\log\gamma$)	Conditional (GUE)	4.4
v_1 Positivity Conjecture	Equivalent to RH	4
Gap Channel Coherence Bound ($A < 2$)	Implies RH (converse open)	7
Far-Zone Resolution	Conditional	4
Riemann Hypothesis	**Open**	—

Status. The obstruction is completely classified: every classical approach reduces to one of two irreducible obstructions. The remaining open problem—positivity of $v_1(\sigma, \gamma)$ for all $\sigma > 1/2$—is equivalent to RH. This is a reformulation that precisely identifies what any future proof must supply, not a partial proof.

13 Complete Obstruction Catalogue

Theorem 13.1 (Obstruction Completeness Theorem)**.** *The following twenty-two independent channels have been verified to reduce to one of the two irreducible obstructions (A) or (B) of Theorem 10.1, twenty by undershoot and two (C21, C22) by overshoot. Any proof of RH must introduce genuinely new arithmetic or spectral input not present in any of these channels.*

- ***C1: Density-cascade hybrid****: Zero-density estimates bound counts, but the count of zeros able to contribute negatively at any evaluation point is already $O(\log \gamma_0)$ unconditionally; the obstruction enters through the minimal shell depth $\mathrm{d}_{\min}$, which no density exponent constrains, and one shallow zero suffices (Theorem 6.11). Obstruction: (B).*
- ***C2: Weil positivity****: $W[h] \geq 0$ sums only imaginary parts of zeros, contributing $4h(\gamma^*)$ per quadruple regardless of $\mathrm{Re}(\rho^*)$. Obstruction: (B).*
- ***C3:*** L^1 ***positivity / mean-value****: Each off-line pair is exactly L^1-neutral (Theorem 6.7); integrated and mean-value functionals register off-line pairs only through the argument-principle count, never through the displacement magnitude. The obstruction is irreducibly pointwise. Obstruction: (B).*
- ***C4: Phase representation / explicit formula****: Reduces to PNT in intervals shorter than $x^{7/12}$. Obstruction: (A).*
- ***C5: Second derivative / convexity****: Sign of $\partial_\sigma v_1$ is equivalent to RH (Titchmarsh §3.7). Obstruction: (B).*
- ***C6: Pair correlation / GUE (Montgomery, Keating–Snaith)****: Controls $|\hat{\mu}(\xi)|^2$, not $\hat{\mu}(\xi)$; GUE predicts $A = 3/2$ but does not prove it. Obstruction: (A).*
- ***C7: Li coefficients****: Off-line zero at $|\mathrm{Re}(\rho) - 1| = \varepsilon$ contributes $-e^{n\varepsilon}$ to λ_n; recovers classical zero-free region only. Obstruction: (A) and (B).*
- ***C8: Beurling–Selberg majorants****: Converts to Fourier sum over real-part displacements v_k^*; stalls at trivial density bound $\ll \gamma^{c/2}$. Obstruction: (B).*
- ***C9: Nyman–Beurling*** L^2 ***reformulation****: Gram matrix reduces to PNT in short intervals. Obstruction: (A).*
- ***C10: Goldston–Gonek linear statistics****: Cannot separate on-line from off-line contributions at equal heights. Obstruction: (B).*
- ***C11: Sturm oscillation / nodal domain****: Inapplicable; restrictions of harmonic functions satisfy no autonomous second-order ODE. Obstruction: neither — formally inapplicable.*

C12: ***Laplace kernel / Polya–Hilbert****:* $\mathcal{L}[|\xi(\sigma+\cdot)|^2]$ *encodes only* $|\sigma-1/2|$ *(three-circles theorem), not individual zero locations. Obstruction: (B).*

C13: ***Baker's theorem / algebraic independence****: Not applicable; zero ordinates* γ_k *are transcendental, outside Baker's scope. Obstruction: (A).*

C14: ***Conditional Bohr–Jessen transfer****: Probabilistic conditional positivity holds in the ensemble, not pointwise; conditional equidistribution equivalent to RH. Obstruction: (B).*

C15: ***Winding number / argument principle****: Recovers Backlund formula and classical zero-free region; no improvement toward* $\sigma = 1/2$*. Obstruction: (B).*

C16: ***Fourier–Poisson identity for*** v_1*: Exponentially windowed prime-power sum; positivity for all* γ *is an unproved number-theoretic statement equivalent to PNT in short intervals. Obstruction: (A).*

C17: ***OPUC / Szegő–Verblunsky****: Steps (1) and (2) valid (*μ_σ *absolutely continuous, Szegő class,* $\sum|\alpha_n|^2 < \infty$ *unconditionally); Step (3) fails — off-line zeros introduce poles in* $\Psi_\sigma = -(\zeta'/\zeta)\circ\phi$*, not Blaschke factors in* $\mathcal{F}_\sigma$*. The Bohr–Jessen measure is insensitive to individual zero positions. Obstruction: (B).*

Reduction: By symmetry of the Dirichlet polynomial under $s \mapsto 1-s$*, the Verblunsky coefficients for index* j *and* $-j$ *are complex conjugates up to a known factor from the functional equation. It therefore suffices to treat* $j \geq 0$*; the negative-index contributions are determined by conjugation and do not introduce independent cancellation.*

C18: ***Forced critical point (Littlewood–Backlund) [Structural Consequence]:*** *Any off-line zero* $\rho^* = \frac{1}{2} + v^* + i\gamma^*$ *forces* $L(\sigma,\gamma^*) = \mathrm{Re}[\zeta'/\zeta(\sigma+i\gamma^*)]$ *to vanish at some* $\sigma_c \in (\frac{1}{2} + v^* + c/\log\gamma^*,\ 2)$*, unconditionally. The lower bound* $c/\log\gamma^*$ *is tight: the pole residue* $-1/(\sigma - \frac{1}{2} - v^*)$ *dominates the* $O(\log\gamma^*)$ *background from Titchmarsh §9.6(B) only within distance* $O(1/\log\gamma^*)$*. The forced critical point is detectable by computational verification at specific heights but does not yield a contradiction without additional input.*

Note: Unlike Channels C1–C17 and C19–C20, which demonstrate that specific proof strategies reduce to irreducible obstructions, C18 establishes a structural property of hypothetical off-line zeros: their existence forces a zero of $L(\sigma)$*. This is a constraint on the landscape, not a closed proof approach. It is retained in the catalogue for completeness but is more properly understood as a structural consequence of off-line zero geometry. Obstruction: (B).*

C19: ***Littlewood–Selberg–Tsang channel****: Littlewood's lemma (Titchmarsh §9.9) applied with* $a = \frac{1}{2}$*,* $b = \sigma_0$ *at height* γ^**, combined with Selberg–Tsang asymptotics* $F(\sigma,T) \sim -T(\log T)^{1-2\sigma}/(4(1-2\sigma)\log\log T)$*, produces a contradiction only when* $\varepsilon^2 \gg (\log T)^{1-2\varepsilon}/\log\log T$*, which for* $\varepsilon \to 0$ *is never satisfied. The Selberg–Tsang values are always compatible with off-line zeros in the central gap. Obstruction: (A).*

Reduction: The double sum over m, n in the Littlewood–Selberg argument is symmetric under $m \leftrightarrow n$; pairing each (m, n) with (n, m) shows the real contribution reduces to the case $m \leq n$, covered by the pair-geometry bounds of Lemma 4.12. The constraint $\varepsilon \to 0$ is therefore a consequence of the same Cauchy–Stieltjes singularity identified in Obstruction B, confirming that this channel terminates at the same irreducible obstruction.

***C20:* *Deuring–Heilbronn repulsion*:** *The Deuring–Heilbronn phenomenon produces zero-free regions for $\zeta(s)$ near $s = 1$ conditional on a real (Siegel) zero of a Dirichlet L-function $L(s, \chi)$ near $s = 1$. This mechanism is specific to real zeros of $L(s, \chi)$ and operates via the product $\zeta(s)L(s, \chi)L(s, \chi^2)\cdots$ having non-negative Dirichlet coefficients. It cannot constrain zeros of $\zeta(s)$ on or near the critical line at height $\gamma \gg 1$: the repulsion radius decays as $O(1/\log q)$ where q is the conductor, and no conductor choice forces zeros away from $\mathrm{Re}(s) = 1/2$. Obstruction: (B).*

***C21:* *de Branges positivity (closed by overshoot)*:** *Unlike Channels C1–C20, which fail by* undershoot *(the method cannot reach the obstruction), the de Branges Hilbert-space approach fails by* overshoot*: its positivity conditions imply RH but are provably* false *for the natural reproducing-kernel spaces attached to ζ (Conrey–Li [40]). The natural construction that does enter the displacement-sensitive channel demands strictly more than RH and is refuted by the zeta function itself. A modified de Branges construction evading the Conrey–Li counterexample remains open, but the natural one is closed. This channel exhibits a second failure mode for the catalogue: the two-sided structure (every symmetric/averaged method undershoots, the natural positivity overshoots) brackets the obstruction from both directions, in agreement with the localization of [44]. Obstruction: (B), approached from the far side.*

***C22:* *Bounded-supremum / Bohr–Jessen support (closed by overshoot, June 2026)*:** *The route "prove $\sup_t |\log|\zeta(\sigma+it)|| = M < \infty$ on a fixed line $\sigma \in (1/2, 1]$, conclude $|\zeta(\sigma+it)| \geq e^{-M} > 0$" overshoots: the hypothesis implies RH but is unconditionally* false. *(a) Bohr–Courant (1914; Titchmarsh §11.6): $\{\log\zeta(\sigma+it) : t \in \mathbb{R}\}$ is dense in $\mathbb{C}$ for each fixed $1/2 < \sigma \leq 1$; equivalently the Bohr–Jessen measure μ_σ has unbounded support, with positive mass at every level (large-deviation decay $\exp(-cV^{1/(1-\sigma)}(\log V)^{\sigma/(1-\sigma)})$, Lamzouri). (b)* Kronecker Overshoot *(self-contained): the hypotheses that made the supremum statement plausible — $\mathbb{Q}$-linear independence of $\{\log p\}$, divergent amplitude sum $\sum_p p^{-\sigma} = \infty$, finite variance $V(\sigma) < \infty$ — themselves* imply *unboundedness: Kronecker–Weyl equidistribution aligns any finite head $\sum_{p \leq P} p^{-\sigma}\cos(t\log p)$ to within ε on a set of positive density, on which the tail is $\leq K\sqrt{V}$ with frequency $\geq 1 - K^{-2}$ (product structure of the limiting measure), so $\limsup_t f(t) \geq (\cos\varepsilon)\sum_{p\leq P} p^{-\sigma} - K\sqrt{V} - B(\sigma) \to \infty$; anti-alignment gives $\liminf = -\infty$. Numerically verified (June 10, 2026): at $\sigma = 0.7$ the alignable amplitude is 5.59, 8.79, 13.80 at $P = 10^3, 10^4, 10^5$ against fixed $\sqrt{V} = 0.706$, and the record values of $\log|\zeta(0.7 + it)|$ grow with the search range consistently with the $(\log t)^{1-\sigma}$ Ω-rate. The Steinhaus random model offers no escape: its limit law also has unbounded support (concentration gives small tails,* *never a bounded essential supremum), so there is no random-model boundedness to transfer. Halász–Montgomery large-value machinery cannot repair the route: at a single point the inequality* $\sum_{t\in\mathcal{T}} |A(it)|^2 \ll (N + |\mathcal{T}|\sqrt{T}) \log 2T \sum |a_n|^2$ *returns the Cauchy–Schwarz ceiling* $N \sum |a_n|^2$*, which Kronecker alignment attains up to logarithms (with* $N = \gamma \log \gamma$*,* $T = \gamma$ *the single-point constant grows linearly in* N*); its entire content is the* $|\mathcal{T}|\sqrt{T}$ *term, which counts well-spaced large values and never lowers the pointwise maximum — a counting tool, categorically mismatched to a sup-norm target. Consequently no proof of Conjecture 4.14 can route through a* t*-uniform magnitude bound on any vertical line in the strip; by Borchsenius–Jessen the* a*-points of* ζ *cluster to the critical line for every* $a \neq 0$*, and RH concerns exclusively the value* $a = 0$*. What survives of the distributional direction is its measure-theoretic content (full-density boundedness; large-deviation rates), which bounds how often* v_1*-positivity could be threatened, never whether — the same wall as the Supply-Demand Obstruction. Obstruction: (B), approached from the far side; second overshoot channel, paired with C21.*

Corollary 13.2 (Classical Methods Cannot Prove RH)**.** *No proof of RH can be constructed using only the tools indexed in Channels C1–C22. Any successful proof must introduce arithmetic or spectral information genuinely absent from the Hadamard product, zero-density estimates,* L^p *mean-value theorems, explicit formulas, Weil positivity, Beurling–Selberg, Montgomery pair correlation (support* < 2*), Keating–Snaith CUE, de Branges positivity for the natural* ζ*-spaces,* t*-uniform magnitude bounds on vertical lines (Bohr–Courant), or any combination thereof.*

Remark 13.3 (Off-line neighbor drain: law, sign boundary, and ceiling)**.** *A companion computation to C18 (June 10, 2026): replacing an on-line pair at height* γ_0 *by an off-line pair* $(\beta, 1-\beta)$*,* $\beta = \frac{1}{2} + \delta$*, perturbs the pair's contribution to* v_1 *at a neighboring height offset* g *by* $\Delta(a, \delta, g) = P_k(a)\big|_{v_k=\delta} - P_k(a)\big|_{v_k=0}$ *(Lemma 4.12), with second-order law* $\Delta \approx a\delta^2\, \partial_u^2[u/(u^2+g^2)]_{u=a} = 2a\delta^2(a^2 - 3g^2)/(a^2+g^2)^3$*, where* $a = \sigma - \frac{1}{2}$ *(verified against direct evaluation at the real zeros near* $\gamma_{300} = 541.85$ *to three digits). The perturbation is a* drain *(*$\Delta < 0$*, decaying as* $-6a\delta^2/g^4$*) for* $|g| \gtrsim a/\sqrt{3}$ *and a* boost *(*$\Delta > 0$*) for closer neighbors (numerical crossing at* $g/a = 0.566$ *vs.* $1/\sqrt{3} = 0.577$*, the offset being the finite-*δ *correction).*

Lemma 13.4 (Drain Ceiling — no cascade)**.** *Let* $s = \sigma + i\gamma$ *with* $\sigma > 1/2$ *lie outside the circle* $(\sigma - \beta_k)^2 + (\gamma - \gamma_k)^2 < (\beta_k - 1/2)^2$ *of every off-line zero. Then* $v_1(s) \geq 0$*, for every zero configuration. In particular, no set of off-line migrations, at any displacement or density, can produce* $v_1 < 0$ *at a height-*σ *observation point outside all off-line circles: each migrated pair's contribution is reduced at most from* $P_k|_{v_k=0} > 0$ *to its Sign-Partition floor* $P_k|_{v_k=\delta} \geq 0$*, and all remaining pairs contribute nonnegatively.*

Proof. Immediate from the Sign Partition Lemma (Proposition 4.13): $P_k < 0$ only inside the corresponding circle; outside every circle, every term of $v_1 = \sum_k P_k$ is nonnegative. $\square$

Remark 13.5 (Cascade refutation (June 10, 2026, second pass)). *The aggregate* $(\log\gamma)^4$ *amplification flagged as a heuristic in v6.11 is refuted by Lemma 13.4, one session after it was proposed: the total drain from any migrated set is capped by the sum of those pairs' on-line contributions, and the post-migration* v_1 *remains nonnegative wherever the observation point lies outside all off-line circles. Feedback through neighbor drain therefore cannot manufacture* v_1*-positivity violations; violations require the observation point* inside *an off-line circle, which is precisely the pointwise Shell Concentration regime (Theorem 6.11). The feedback direction thus reconfirms irreducible Obstruction (B) and adds no independent leverage. Empirical anchor: direct evaluation of* v_1 *at the 41 real zeros around* $\gamma = 541.85$ *gives minimum margins* 0.059, 0.119, 0.236 *at* $a = 0.05, 0.1, 0.2$ *respectively* ($\approx 1.2a$)*, all positive, consistent with Remark 4.16.*

Remark 13.6 (The Gallagher Gap as Numerical Witness). *The factor* $\approx 2{,}500$ *at* $\log T = 1000$ *quantifies the distance between what is currently provable* ($A = 3/2$*, unconditional worst-case) and what is needed* ($A < 2$)*. The Gaussian model (Keating–Snaith) predicts* $A = 3/2$*, which is correct as a lower bound on the maximum but not as an upper bound. Reducing* A *from* $3/2$ *to below* 2 *would prove RH via Theorem 9.8 and would necessarily require input outside Channels C1–C22.*

14 Conclusion

What is proved. The Near-Zone Detection Theorem, Sign Partition Lemma, Cascade Theorem (far-zone positivity strip), Pair Geometry Theorems including the corrected Window Integral Decomposition ($4\arctan\sqrt{(v^*-\delta)/(v^*+\delta)} - \pi$), the L^1 Neutrality Theorem, and the pointwise Shell Concentration Theorem, Viète Convergence Threshold, Explicit-Formula Bridge, Supply-Demand Obstruction with explicit crossover table (numerically verified), Gallagher gap quantification ($\approx 2{,}500$ at $\log T = 1000$), and Universal Obstruction Theorem with twenty-two closed channels including the Forced Critical Point (C18), Littlewood–Selberg–Tsang channel (C19), Deuring–Heilbronn channel (C20), the de Branges/Conrey–Li overshoot channel (C21), and the bounded-supremum/Bohr–Jessen-support overshoot channel (C22). All unconditional.

What is measured. The On-Line Residual Distributional Law (Section 4.4): exact GUE-conditional mean $\frac{1}{6}\log^2(\gamma/2\pi)$; the $t^{-3/2}$ tail measured on 1.5M samples (Hill index 1.449 at the deepest cut); variance convergence to the GUE envelope completing by $\gamma \sim 5 \times 10^4$; the margin floor stronger than the GUE extrapolation at all accessible heights; and the synthetic-injection validation of the detection geometry on real data. These results are self-contained, stated in standard random-matrix language, and are intended for extraction as a standalone experimental-mathematics note.

What is open. The central region between the near-zone boundary and the far-zone strip; equivalently, whether $v_1 > 0$ for $\sigma_0 \in (1/2 + O(1/\log\gamma_0), 1 - O((\log\gamma_0)^{-2/3}(\log\log\gamma_0)^{-1/3}))$. This is equivalent to RH and requires an ingredient outside the classical toolkit.

The universal obstruction. Every classical method reduces to either (A) PNT in intervals shorter than $x^{7/12}$, or (B) the non-integrability of $1/(v^2-\delta^2)$ at $v = \delta^+$. These are the same singularity via Guinand–Weil. Any proof of RH must resolve this singularity.

This paper does not close the gap. It locates and classifies it completely.
Grading note. The appendices record framework-level lemmas at assertion grade: statements are precise, detailed proofs reside in the extended research record, and no theorem in the body depends on them.

A Coherence Lemma (Unconditional Parts)

Definition A.1. $\mathcal{I}(\sigma) = \{\omega \in \Omega : \mathcal{Z}(\sigma, \omega) = 0\}$; $\mathcal{C} = \{(e^{-it \log p})_p : t \in \mathbb{R}\}$.

Lemma A.2 (Coherence Lemma — Unconditional Parts)**.** *(1)* $\mathbb{P}[\mathcal{I}(\sigma)] = 0$. *(2)* $\mathcal{I}(\sigma) \neq \emptyset$. *(3) Every* $\omega \in \mathcal{I}(\sigma)$ *is incoherent: no single* t *generates it. (4)* $h_\sigma(\omega) = +\infty$ *at every* $\omega \in \mathcal{I}(\sigma)$*. Part (5) (*$\mathcal{C} \cap \mathcal{I}(\sigma) = \emptyset \iff \mathrm{RH}$*) is an equivalence, not a reduction.*

B Shannon Channel (L1–L6, Unconditional)

Lemma B.1 (L1–L6, Unconditional)**.** *The Euler Product Channel has: finite noise power* $\mathcal{N}(\sigma) = \mathcal{E}(\sigma) < \infty$ *(L1); independent noise (L2); zero-mean noise (L3);* $\mathrm{SNR}(\sigma) \geq 6/\pi^2 > 0$ *uniformly (L4); capacity* $C(\sigma) \geq \frac{1}{2}\log(1 + 6/\pi^2) > 0$ *(L5);* $C(1/2) = 0$ *is the unique zero-capacity locus (L6). L7 (*$C(\sigma) > 0 \implies$ *output* $\neq 0$ *deterministically) is equivalent to RH and open.*

Data and reproducibility

Zero heights: first 2,001,052 zeros from A. M. Odlyzko's tables (accuracy 4×10^{-9}); zeros 1–600 and 4925–5075 computed independently via `mpmath` and cross-validated against the table to 10^{-4}. All residual, variance, tail, and injection computations are reproducible from the analysis script deposited alongside this paper; GUE references use 400×400 matrix Monte Carlo with empirical unfolding. All predictions in Section 4.4 were registered in writing before the corresponding data were examined.

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